Preface

FLUID MECHANICS is that study of fluid motion involving a rational method of approach based on general physical laws and consistent with the results of modern experimental study. The word *fluid* implies a treatment of both liquids and gases. Fluid mechanics uses the same principles employed in the mechanics of solids. The modern trend is to avoid a collection of specialized empirical data of limited applicability; the trend is to develop general relations, and to organize experimental observations in a form suitable for use over a wide range of conditions.

It is difficult to find a branch of engineering which is not concerned with fluids or which does not make use of fluids. As Dr Hugh L. Dryden, one of the prominent workers in fluid mechanics, states, "Fluid mechanics is one of the basic sciences which finds practical application in such diverse fields as aeronautics, hydraulic engineering, ventilation, chemical engineering, sanitary engineering, marine engineering, automotive engineering, railroad engineering, turbine design, and ballistics."[1] To this list might be added mechanical engineering, acoustics, meteorology, and a host of other applications. At first thought it might appear that the flow phenomena observed in these and other fields have little in common. Closer examination, however, reveals that the complicated flow phenomena involved have a common foundation in the principles of fluid mechanics. The basic equilibrium and dynamical equations have the same form for all fluids. In numerical cases, of course, due allowance must be made for the different properties of the fluids.

There is real economy and value in studying at one time the same principles underlying the flow of different fluids. Such a study tends to develop a sound background and to make one versatile in approaching problems new to him. Studying fluid mechanics is analogous to studying applied mechanics and thermodynamics as a broad, unified preparation for subsequent specialized courses.

The aim of this book is to present an introduction to the fundamentals of fluid mechanics. The wisdom of concentrating on all

[1] *The Rôle of Transition from Laminar to Turbulent Flow in Fluid Mechanics*, by H. L. Dryden, in the book *Fluid Mechanics and Statistical Methods in Engineering*. University of Pennsylvania Press, Philadelphia, 1941, page 1.

the minute details of all existing applications is open to som question, in view of the time usually available in a first course and the possible development of new and unexpected applications.

A serious attempt has been made to provide a balanced, practical treatment in a logical fashion, and to keep physical concepts and basic established quantitative relations in the foreground. Physical concepts are stressed with the hope that once the student has a good physical picture he can proceed of his own accord, with interest, in understanding and analyzing flow phenomena. As a simple example of the tendency in handling quantitative relations, Chapter 4 discusses the energy equation first in its general form. Special cases are subsequently worked out as occasion requires.

The plan of the book follows in some respects the logical sequence of rigid-body mechanics—first statics, then kinematics and dynamics. The writer feels that the powerful tools of dimensional analysis and dynamic similarity should be introduced as soon as possible, and employed as much as is feasible. Viscosity however, should be covered first in order to provide a satisfactoı treatment of these two topics. This outlook explains, to som extent, the arrangement of the first six chapters and the genera treatment in the following chapters. After covering foundation material in the first six chapters, particular cases of flow are considered, incompressible flow *in* pipes and channels, and flow *around* bodies. Several chapters discuss compressible flow. The isolation of some of the later chapters may be helpful in providing flexibility in arranging a course. For example, if the subject o lubrication is covered in a machine-design course, then Chapter 15, on lubrication, may be condensed or omitted without causing a serious break. The last chapter gives an elementary discussion of hydrodynamics; this chapter may be presented earlier if so desired.

Experience has shown that problem work on the part of the student is very helpful, if not necessary, to give him a working knowledge of the subject. Problem work offers the student a definite test and challenge to supplement his reading.

It is not possible to give adequate, explicit credit to all those individuals and organizations who have directly and indirectly given aid in the preparation of this book. However, the writer is deeply grateful to all of them for their help, and wishes to acknowledge it as best he can. Many fellow workers, in industrial and research work, have made suggestions, particularly as to material content. Various instructors and students in different schools have offered suggestions as to method of presentation.

The author is particularly indebted to some staff members of Purdue University. Dean A. A. Potter has given advice regarding the development of course work in fluid mechanics which is reflected in this book. Professor H. L. Solberg has made many pertinent suggestions as to subject matter, arrangement of topics, and method of presentation. Professor K. D. Wood has contributed helpful ideas regarding the preparation of the manuscript.

R. C. Binder

Contents

CHAPTER 1

Introduction—Some Fluid Properties

1. Fluids

A fluid is a substance which when in static equilibrium cannot sustain tangential or shear forces. Fluids yield continuously to tangential forces, no matter how small. This property distinguishes between two states of matter, the solid and the fluid. Imagine two plates of metal joined by a solid rivet, as shown in Fig. 1. For small pulls, the solid rivet sustains shear forces in static equilibrium. If a fluid, such as oil, water, tar, or air, were substituted for the solid rivet, there would be continuous relative motion of the plates, even for infinitesimal pulls. Real fluids in motion sustain shear forces because of the property of viscosity. Viscosity will be discussed in detail in Chapter 5.

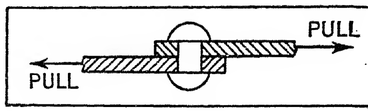


Fig. 1. Plates joined by a rivet.

Fluids are commonly divided into two subclasses: liquids and gases. A liquid occupies a definite volume, independent of the dimensions of the vessel in which it is contained. A liquid can have a free surface, like the surface of a lake. A gas, on the other hand, tends to expand to fill any container in which it is placed. Sometimes a distinction is drawn between gases and vapors. A vapor, such as steam or ammonia, differs from a gas by being readily condensable to a liquid.

Gases are frequently regarded as compressible, liquids as incompressible. Strictly speaking, all fluids are compressible to some extent. Although air is usually treated as a compressible fluid, there are some cases of flow in which the pressure and density changes are so small that the air may be assumed to be incompressible. Illustrations are the flow of air in ventilating systems and the flow of air around aircraft at low speeds. Liquids, like oil and water, may be considered as incompressible in many cases; in other cases, the compressibility of such liquids is important. For instance, common experience shows that sound waves travel through water and other liquids; such pressure waves depend upon the compressibility or elasticity of the liquid.

Fluids will be discussed as continuous media. Actually, liquids and gases consist of molecules and atoms; fluid properties and phenomena are intimately related to molecular behavior. In most engineering problems, however, the mean free path of the molecule is small in comparison with

the distances involved, and the flow phenomena can be studied by reference to bulk properties, without a detailed consideration of the behavior of the molecules.

2. Pressure

Shear force, tensile force, and compressive force are the three kinds of force to which any body may be subjected. Fluids move continuously under the action of shear or tangential forces. Fluids will support tensile forces to the extent of the cohesive forces between the molecules. Since such forces are very small, it is common practice in engineering problems to assume that a fluid cannot sustain a tensile stress. It is well established that fluids are capable of withstanding a compressive stress, which is usually called pressure.

The term *pressure* will be used to denote a *force per unit area.* Probably a more descriptive label would be "pressure-intensity," but the briefer term pressure will be employed. Sometimes the term pressure is used in the sense of a total force, but this practice in fluid mechanics is apt to be confusing, and will not be followed.

Atmospheric pressure is the force exerted on a unit area owing to the weight of the atmosphere. Many pressure-measuring instruments indicate relative or *gage* pressure. Gage pressure is the difference between the pressure of the fluid measured and atmospheric pressure. *Absolute* pressure is the sum of gage pressure plus atmospheric pressure. The word *vacuum* is frequently used in referring to pressures below atmospheric.

3. Specific weight and density

This book will maintain a definite distinction between force and mass, between specific weight and density, because such distinctions are essential for a correct and clear account. Weight is not the same as mass. A body of mass m is attracted towards the center of the earth with a force of magnitude mg, where g is the gravitational acceleration, or the acceleration of a freely falling body. The force mg is termed the weight of the body of mass m. The weight of a given mass changes when the gravitational acceleration changes.

A consistent set of units, following modern practice, will be employed in order to avoid confusion. When using the American system, the pound will be taken as the unit of force, and the slug as the unit of mass. This terminology follows the practice in rigid-body mechanics. Specific weight w is defined as weight per unit volume. Specific volume v is defined as volume per unit weight, and is the reciprocal of specific weight. Density ρ (Greek letter rho) is defined as *mass* per unit volume. Since force equals mass times acceleration,

$$w = \rho g \qquad \text{or} \qquad \rho = \frac{w}{g}.$$

For example, if the specific weight of water is 62.42 pounds per cubic foot (at a certain temperature and pressure), and g is taken as 32.174 feet per second per second, then the density is 62.42/32.174, or 1.94 slugs per cubic foot. Figure 2 shows the variation of the specific weight of water with temperature at atmospheric pressure. Precise calculations may require a detailed consideration of measured values at different temperatures and pressures. Such data may be found in various extensive refer-

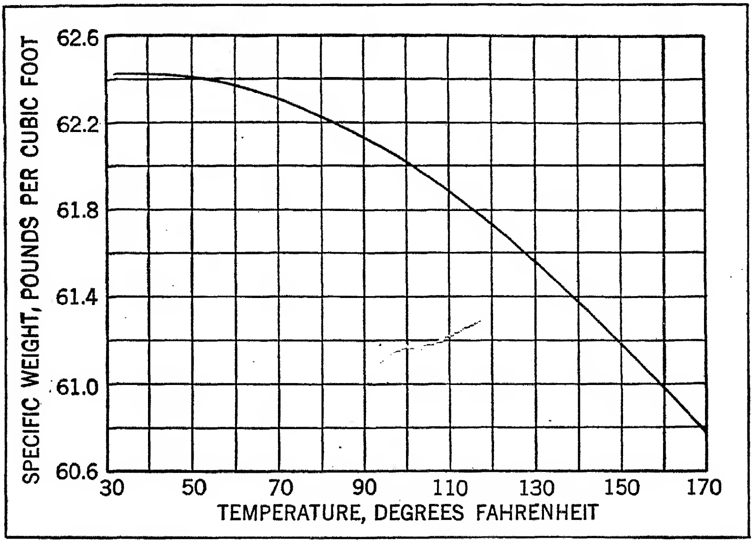


FIG. 2. Specific weight of pure water at atmospheric pressure.

ence works.[1] Unless otherwise specified, the subsequent discussions and problem work in this text will take g as 32.2 feet per second per second, and the specific weight of water as 62.4 pounds per cubic foot. In the metric system, the dyne is a unit of force, the gram is a unit of mass, and g is about 981 centimeters per second per second.

4. Specific weight of liquids

The specific weight of a substance can be calculated if the specific gravity is known. The specific gravity of a substance is defined as the ratio of its specific weight (or density) to the specific weight (or density) of some standard substance. For liquids the standard usually employed is

[1] See, for example, *Smithsonian Physical Tables*, Smithsonian Institution, Washington, D. C.; *International Critical Tables*, McGraw-Hill Book Co., Inc., New York, N. Y.; and *Handbook of Chemistry and Physics*, Chemical Rubber Publishing Co., Cleveland, Ohio.

either water at 4° centigrade (39.2° Fahrenheit), or water at 60° Fahrenheit. The mass in grams of 1 milliliter of water at 4° centigrade is unity. Thus, for practical problems the specific gravity of a liquid is frequently taken as numerically equal (not dimensionally equal) to the density in grams per milliliter. The specific gravities of some liquids are listed in Table 1. The densities of some common lubricants are included in the chapter on lubrication, Chapter 15.

TABLE 1[2]
SPECIFIC GRAVITIES OF SOME COMMON LIQUIDS
(Referred to water at 39.2° Fahrenheit)

	Specific gravity	Temperature, degrees Fahrenheit
Alcohol, ethyl......	0.807	32
Benzene...........	0.899	32
Gasoline...........	0.66–0.69	
Glycerine..........	1.260	32
Mercury...........	13.546	68
Oil, castor.........	0.969	59
Oil, linseed (boiled)	0.942	59
Turpentine........	0.873	60.8

5. Equation of state for gases

The specific weight of a gas can be calculated from the *equation of state*, or pressure-volume-temperature relation. The "ideal" or "perfect" gas law is commonly employed.[3] If v is specific volume, p is absolute pressure, and T is absolute temperature, this simple equation of state is

$$pv = RT \tag{1}$$

where R is the gas constant. On the Fahrenheit scale, $T = t + 459.69 = t + 460$ (approximately), where t is the usual thermometer reading. If p is expressed in pounds per square foot, v in cubic feet per pound, and T in degrees Fahrenheit, R has the dimensions of feet per degree Fahrenheit. Some average values of R are listed in Table 2.

Application of Avogadro's principle, that "all gases at the same pressures and temperatures have the same number of molecules per unit volume," indicates that the product of molecular weight M and the gas constant R is the same for all gases. Table 2 shows some variation. The

[2] *Smithsonian Physical Tables*, Eighth Revised Edition, vol. 88, Smithsonian Institution, Washington, D.C., 1933.

[3] In thermodynamics the term "ideal" or "perfect" gas is sometimes defined as one which obeys the relation $pv = RT$. In hydrodynamics an "ideal" or "perfect" fluid is sometimes defined as one which is frictionless. Although this book will avoid this double use, the distinction should be noted in reading current literature.

product MR is termed the "universal gas constant"; a convenient approximate value is 1544. For practical purposes (not requiring a high degree of precision) R can be taken as 1544 divided by the molecular weight of the gas. The molecular weight is easily found from known atomic weights.

TABLE 2
SOME AVERAGE FACTORS FOR GASES

	R feet per degree Fahrenheit absolute	MR	k
Air	53.3	1545	1.40
Carbon dioxide	34.9	1536	1.30
Carbon monoxide	55.1	1543	1.40
Helium	386.	1544	1.66
Hydrogen	767.	1546	1.40
Methane	96.2	1539	1.31
Oxygen	48.25	1544	1.40

For real gases, Equation (1) is accurate at ordinary temperatures and for relatively small changes in pressure or volume. No gas obeys this simple relation in the region of liquefaction. Suitable vapor tables and charts should be consulted for such problems.

6. Gas processes

If a mass of gas undergoes changes in pressure and volume, some information about the process is necessary in order to specify the pressure-volume relation at each step. For an isothermal or constant temperature process the equation of state gives

$$pv = \text{constant.}$$

The general, or "polytropic," process is given by the equation

$$pv^n = \text{constant}$$

where the exponent n is some general factor. $n = 1$ for the isothermal case. For the expansion or compression process in an engine or a sound wave, n may be 1.4 or some other value different from 1.

An *adiabatic* process is one in which no heat (thermal energy in transition) is added to or removed from the fluid mass. Energy considerations[4] and application of the equation of state show that $n = k$ for an adiabatic process, where

$$k = \frac{\text{specific heat at constant pressure}}{\text{specific heat at constant volume}}.$$

Representative values of k are listed in Table 2.

[4] See Chapter 4.

SELECTED REFERENCES

The Physics of Solids and Fluids by P. P. Ewald, T. Pöschl, and L. Prandtl. Blackie and Son, Ltd., London, 1932, page 153.

Statics by H. Lamb. Cambridge University Press, 1933, page 199.

Applied Thermodynamics by V. M. Faires. Macmillan, New York, 1938.

PROBLEMS

1. What is the specific weight and density of benzene? of mercury?

2. A fluid weighs 25.0 pounds per cubic foot at a location where the gravitational acceleration is 32.17 feet per second per second. What is its density? What is the specific weight of this same fluid at a different location where the gravitational acceleration is 32.13 feet per second per second?

3. What is the specific weight and density of carbon dioxide at 90° Fahrenheit and an absolute pressure of 110 pounds per square inch?

4. Calculate the density, specific weight, and specific volume of air at 45° Fahrenheit and 60 pounds per square inch absolute.

5. What is the gas constant R for air with pressure in pounds per square foot, specific volume in cubic feet per pound, and temperature in degrees centigrade?

6. A submarine is closed at the surface, with the air content at 80° Fahrenheit and 14.7 pounds per square inch absolute. After submerging, the air temperature drops to 58° Fahrenheit. What is the air pressure for the submerged conditions if the hull has not changed in size?

7. Hydrogen, initially at 40 pounds per square inch absolute and 60° Fahrenheit, expands isothermally to 15 pounds per square inch absolute. What is the final specific volume? What is the final specific volume if the expansion is adiabatic?

CHAPTER 2

Fluid Statics

> A solid heavier than a fluid will, if placed in it, descend to the bottom of the fluid, and the solid will, when weighed in the fluid, be lighter than its true weight by the weight of the fluid displaced—ARCHIMEDES.[1]

A wide variety of problems involves fluids in static equilibrium. Such problems can be analyzed by basic concepts and relations which are relatively few and straightforward.

7. Nature of static fluid

A body of fluid in static equilibrium is free in every part from tangential or shear forces. Examples are a fluid at rest in a container and a body of fluid in a container moving with uniform velocity. One fluid layer does not slide relative to an adjacent layer; there is no distortion of the fluid elements. Absence of shear means that friction need not be considered. Frictional effects arise only when there is relative sliding or shearing, as with a fluid in motion. Since shear and tensile forces are absent in a static fluid, the only forces involved are compressive.

Pascal's important law, that the pressure in a static fluid is the same in all directions, can be shown by studying the forces acting on any infinitesimal element in a body of fluid. In Fig. 3 the distance between the two triangular faces is unity, and the sides dx and dy are infinitesimal. Each of the pressures p_1, p_2, and p_3 is normal to the face upon which it acts. Since the forces balance in equilibrium, taking vertical and horizontal components of the forces gives

$$p_2 dx - p_1 ds \cos\theta = 0, \qquad p_3 dy - p_1 ds \sin\theta = 0.$$

As $dx = ds \cos\theta$ and $dy = ds \sin\theta$,

$$p_2 - p_1 = 0, \qquad p_3 - p_1 = 0, \qquad p_1 = p_2 = p_3.$$

The pressure at a point in a static fluid is the same in all directions. This result is different from that obtained for a stressed solid in static equilibrium; in such a solid the stress on a plane depends on the orientation of that plane. The preceding analysis can be extended to show that

[1] As translated by T. L. Heath in his book, *The Works of Archimedes*, Cambridge University Press, 1897.

the pressure at a point within a fluid is the same for any state of motion provided there is no shear stress (as with a frictionless fluid).

It is common to refer to a liquid surface in contact with the atmosphere as a "free" surface. From one point of view, if a liquid has a free surface,

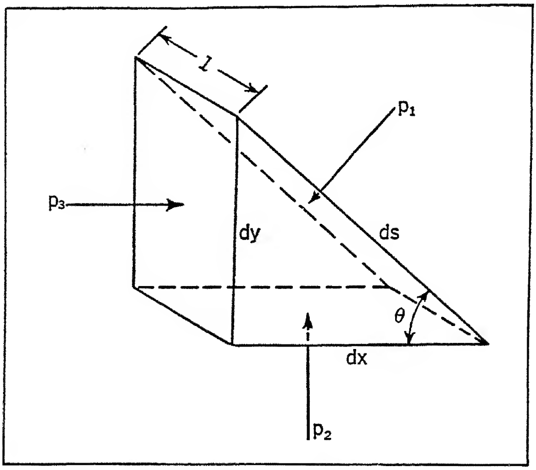


Fig. 3. Equilibrium of a small fluid element.

then there is *no* pressure upon that surface. In general, however, a surface upon which only atmospheric pressure acts is called a free surface. A static liquid has a horizontal free surface if gravity is the only force acting. Imagine a surface other than horizontal. Shear forces would be necessary in order to hold one point on the surface above another. A sloping surface may exist as a transient flowing condition, but when static equilibrium is reached, the free surface of the liquid is horizontal.

8. Fundamental equation of fluid statics

Figure 4*a* shows an infinitesimal element in the body of any static fluid. Figure 4*b* shows the element in detail. z is a vertical distance measured positively upward, and dA is an infinitesimal area. p is the pressure on the top surface, and $(p + dp)$ is the pressure on the bottom surface, the pressure increase being due solely to the fluid weight. The weight of the element is $wdzdA$. Balancing forces on the free body element in the vertical direction gives

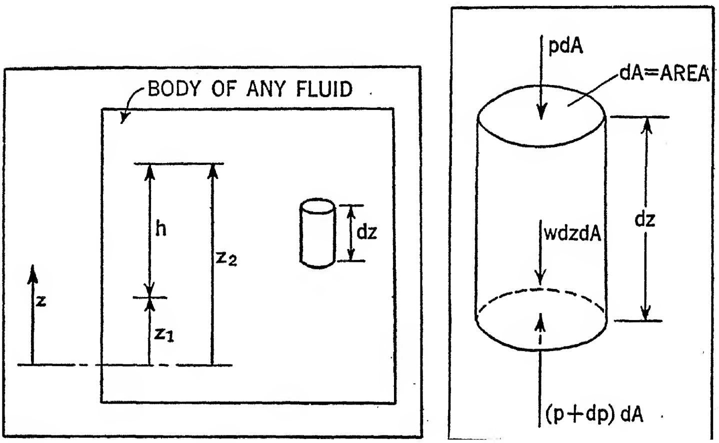
$$dpdA = -wdzdA \qquad \text{or} \qquad dp = -wdz. \tag{2}$$

Since z is measured positively upward, the minus sign indicates that the pressure *decreases* with an *increase* in height. *Equation* (2) *is the funda-*

mental equation of fluid statics. It states that the pressure decreases in the upward direction, the decrement per unit length being equal to the weight per unit volume. dp is zero when dz is zero; the pressure is constant over any horizontal plane in a fluid. In integral form Equation (2) becomes

$$\int_1^2 \frac{dp}{w} = -\int_1^2 dz = -(z_2 - \tag{3}$$

The functional relation between pressure and specific weight must be established before Equation (3) can be integrated further. The following



(*a*) (*b*)
FIG. 4. Vertical forces on an infinitesimal element.

articles consider separately the case for incompressible fluids and the case for compressible fluids, together with various practical applications.

9. Pressure-height relation for incompressible fluids

Liquids can be treated as incompressible in many practical cases. In some cases, as for small differences in height, gases can be regarded as incompressible. For an incompressible fluid the specific weight w is constant, and Equation (3) becomes

$$p_2 - p_1 = -w(z_2 - z_1) \qquad \text{or} \qquad \Delta p = wh \tag{4}$$

where Δp is the pressure difference and h is the difference in levels. If w is expressed in pounds per cubic foot, and h in feet, then Δp is in pounds per square foot. h is commonly called a *pressure head,* and may be expressed in feet or inches of water, inches of mercury, or some height of any liquid.

Several pressure and head equivalents for common liquids are convenient for problem solutions. The use of special conversion factors, whose physical significance may be obscure or easily forgotten, is not necessary, if it is kept in mind that standard atmospheric or barometric pressure is 14.7 pounds per square inch, 29.92 inches of mercury, or about 33.9 feet of water. One unit of pressure derived from the barometer is the *atmosphere;* 1 atmosphere equals 14.7 pounds per square inch. A pressure of 4 atmospheres, for example, is $4 \times 14.7 = 58.8$ pounds per square inch.

EXAMPLE. Figure 5 shows a container with oil (specific gravity = 0.90) in static equilibrium. Atmospheric pressure exists at point A. At A the *gage*

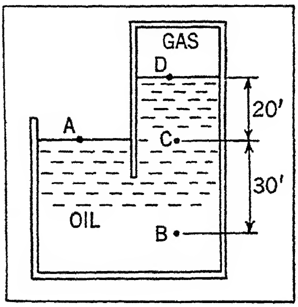


FIG. 5. Container with oil.

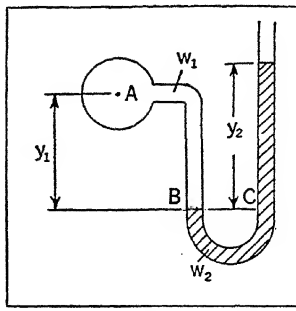


FIG. 6. Open manometer.

pressure is zero, whereas the *absolute* pressure is 14.7 pounds per square inch. The pressure at C equals that at A, because C is at the same level as A. Since

$$-p_C = \frac{0.9(62.4)30}{144} = 11.7 \text{ pounds per square inch,}$$

the pressure at B is 11.7 pounds per square inch gage, or 26.4 pounds per square inch absolute. As

$$p_C - p_D = \frac{0.9(62.4)20}{144} = 7.8 \text{ pounds per square inch,}$$

the absolute pressure at D is $14.7 - 7.8 = 6.9$ pounds per square inch.

10. Manometers

The term *manometer* is applied to a device that measures pressure by balancing the pressure against a column of liquid in static equilibrium. A wide variety of manometers is in use: vertical, inclined, open, differential, and compound. Fundamentally, the use of a manometer is good technique; the instrument is simple, and may be employed for precise pressure measurements. Both the specific weight of a liquid and the height of a liquid column can be measured accurately.

The basic equation for calculating the pressures indicated by liquid manometers is simply $\Delta p = wh$. It is suggested that each manometer installation be worked out as a separate problem, using this basic relation. The blind use of special formulas for special arrangements is not recommended. The student who follows this suggested plan will find it not only sound and instructive, but interesting. A few examples illustrating the method of approach follow.

In the *open* manometer, Fig. 6, the right leg is open to the atmosphere and contains a liquid of specific weight w_2. Consider the problem of determining the pressure at point A in a liquid of specific weight w_1. This liquid partially fills the left leg. Starting with the right leg, the gage pressure at point C is

$$p_C = w_2 y_2.$$

Because the manometer contains a fluid in static equilibrium, the pressure at B equals the pressure at C. Then

$$p_B = p_C = w_2 y_2.$$

The pressure at A is less than that at B:

$$p_B - p_A = w_1 y_1.$$

From the last two relations, the gage pressure at A is

$$p_A = w_2 y_2 - w_1 y_1 \tag{5}$$

As an example, assume that the liquid in the right leg is mercury, that the liquid at A is water, that $y_1 = 5$ feet, and $y_2 = 10$ feet. The pressure at A is

$$\begin{aligned} p_A &= 62.4(13.55)10 - 62.4(5) \\ &= 8140 \text{ pounds per square foot gage,} \end{aligned}$$

or the pressure at A is

$$\tfrac{8140}{144} + 14.7 = 71.2 \text{ pounds per square inch absolute.}$$

The name *piezometer* is sometimes applied to the type of manometer shown in Fig. 6, in which only the liquid at A is used in the U tube. As the name implies, a piezometer is an instrument for measuring pressure.

The *differential* manometer in Fig. 7 is used to measure the pressure difference $p_A - p_B$. For the right leg,

$$p_E = w_2 y_2 + w_3(y_3 - y_2) + p_B.$$

The pressure at C equals the pressure at E. For the left leg,

$$p_C = p_E = w_1 y_1 + p_A.$$

Equating the two foregoing relations gives

$$w_1y_1 + p_A = w_2y_2 + w_3(y_3 - y_2) + p_B,$$
$$p_A - p_B = w_2y_2 + w_3(y_3 - y_2) - w_1y_1. \tag{6}$$

Equation (6) provides a direct calculation of the pressure difference.

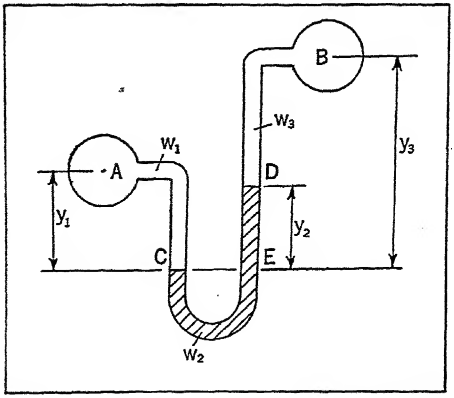


FIG. 7. Differential manometer.

11. Barometers

The widely employed mercurial barometer is essentially a manometer for measuring atmospheric pressure. The glass tube (about one yard long) of the instrument is closed at one end and open at the other. The tube is first filled completely with pure mercury, and then inverted into a vertical position with the open end submerged in a small vessel (cistern) containing mercury. The principle is illustrated in Fig. 8. The direct reading of the barometer gives the height of a column of mercury; the weight of this mercury column balances the weight of the air column above the barometer. Usually the whole unit is enclosed in a brass case for protection, and provided with an adjustable scale and vernier.

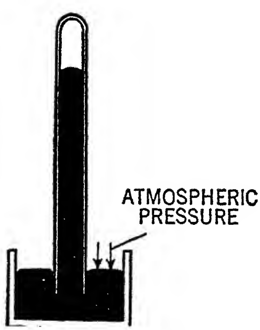
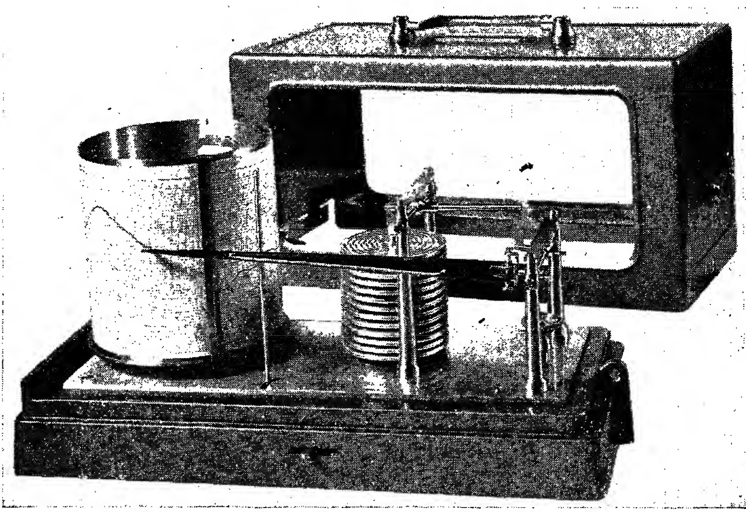


FIG. 8. Main features of the mercurial barometer.

A discussion of the aneroid barometer and the barograph is helpful in bringing out some general features of various pressure-indicating and recording devices. An aneroid barometer is a liquidless instrument for

measuring air pressure. It is more portable and rugged than the mercurial barometer. The usual aneroid barometer has a shallow, air-tight, elastic metal box which expands or contracts according to the atmospheric pressure on the box. The expansion or contraction motion is transmitted by some mechanism to a dial graduated in pressure units. The aneroid barometer must be calibrated and checked with a mercurial barometer.



Courtesy of J. P. Friez & Sons.

FIG. 9. Barograph.

The barograph, as shown in Fig. 9, provides a continuous record of pressure over a period of time. Ordinarily a series of air-tight, corrugated metal shells forms the pressure element. One end of the evacuated metallic box is fixed, whereas the other end moves proportionately to the pressure. This movement is transmitted by a mechanism to a pen which records a trace on a sheet of paper fastened to a revolving drum. The barograph requires checking as far as the absolute value of the pressure is concerned; it is common practice to refer the barograph setting or reading to a mercurial barometer.

12. Bourdon gages

The so-called Bourdon tube is used in various forms of instruments for measuring pressure differences. The essential feature of this type of gage is shown diagrammatically in Fig. 10. The Bourdon tube is a hollow metal tube (brass in many cases), of elliptical cross section, bent in the form of a circle. One end of the Bourdon tube is fixed to the frame at A, whereas the other end B is free to move. The free end

actuates a pointer through a suitable linkage. As pressure inside the tube increases, the elliptical cross section tends to become circular; the tube thus tends to straighten out. A pressure scale or dial can be devised from a calibration of the instrument.

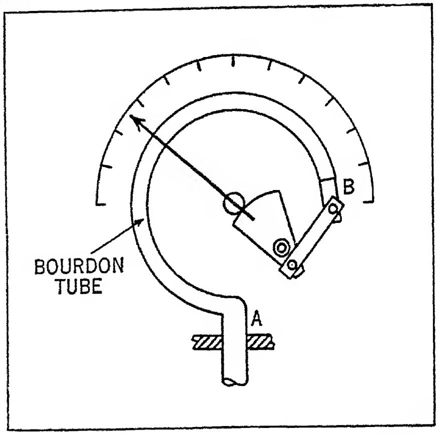


FIG. 10. Essential features of a Bourdon gage.

Note that the position of the free end of the Bourdon tube depends upon the difference in pressure between the inside and the outside of the tube. If the outside of the Bourdon tube is exposed to atmospheric pressure, the instrument responds to *gage* pressure.

Various adaptations of the Bourdon tube are found in current industrial instruments. In some instruments the tube is wound in the form of a spiral or helix with several complete turns. In some cases the Bourdon tube is employed for temperature measurements. The inside of the Bourdon tube is filled with a fluid sensitive to temperature changes. A correlation between temperature and pressure gives a scale reading which can be interpreted in terms of temperature.

13. Total force on plane submerged surfaces

The computation of the magnitude, direction, and location of the total force on a surface acted upon by a fluid is involved in the design of tanks, gates, and other structures and equipment. The force exerted on any elementary surface by a static fluid is perpendicular to that surface. The force on an elementary area dA is pdA. The total force $F = \int pdA$ is perpendicular to the plane.

For gases the pressure is practically constant over areas of usual size; therefore, the total force F on a total area A is

$$F = p\int dA = pA \tag{7}$$

For a horizontal surface in a liquid, as shown in Fig. 11, the pressure is uniform over the surface.

For a nonhorizontal plane surface in a liquid, as shown in Fig. 12, the pressure varies directly with depth according to the fundamental relation $\Delta p = wh$.

In Fig. 13 a submerged plane area BC makes an angle θ with the free surface of the liquid. The problem is to calculate the total force exerted by the liquid on one side of the area. The gage pressure p at depth h is $p = wh$; the force dF on the elementary area dA is

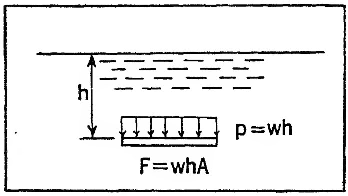


FIG. 11. Horizontal surface in a liquid.

$$dF = pdA = whdA.$$

In terms of y, $dF = wy \sin \theta dA$.

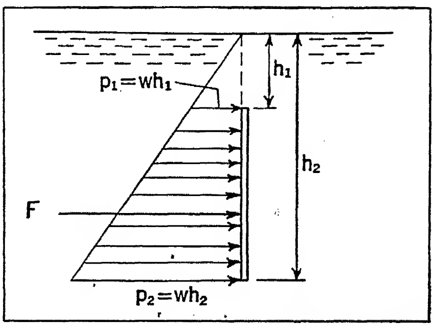


FIG. 12. Vertical surface in a liquid.

The resultant force F is

$$F = \int dF = w \sin \theta \int y dA$$

The centroidal distance $\bar{y}$ is defined by the relation $\bar{y}A = \int y dA$. Since $\bar{h} = \bar{y} \sin \theta$,

$$F = wA\bar{y} \sin \theta = w\bar{h}A. \tag{8}$$

Equation (8) shows that the *total force on a plane area is the product of the pressure at the centroid multiplied by the total area.* Equation (8) provides a means for calculating only the *magnitude* of the total force; the direction of this force is normal to the surface. The location is yet to be determined.

14. Location of total force on a plane area

In Fig. 13 the plane area is acted upon by a system of parallel forces, each normal to the plate. Taking moments about D in the free surface gives

$$Fy_F = \int y dF,$$

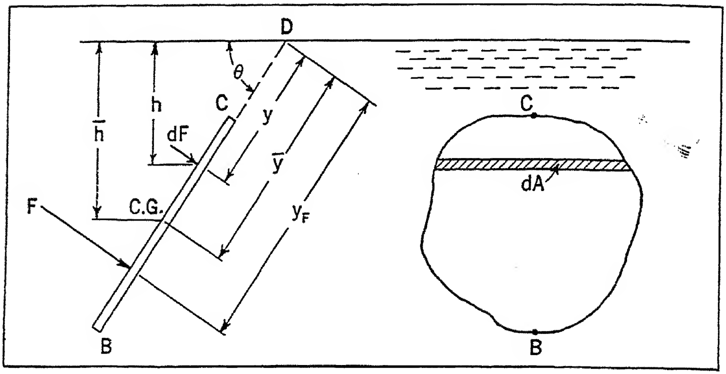


Fig. 13. Total force on a plane area.

where y_F is the distance to the line of action of the total force F. Since

$$F = \int dF = \int wdAy \sin \theta,$$

and

$$= \int ydF = \int wdAy^2 \sin \theta$$

$$y_F = \frac{}{\int ydA} = \frac{I}{A\bar{y}} \tag{9}$$

in which I is the moment of inertia of the area about the axis from which y is measured, and $\bar{y}$ is the centroidal distance. $I = I_G + A\bar{y}^2$, where I_G is the moment of inertia of the area about a parallel axis through the centroid. Then

$$y_F = \frac{I_G + A\bar{y}^2}{A\bar{y}} = \bar{y} + \frac{I_G}{A\bar{y}}. \tag{10}$$

Equation (10) shows that the line of action of the resultant force is always *below* the centroid by the distance $I_G/A\bar{y}$. As the radius of gyration k of an area is defined as equal to $\sqrt{I/A}$, Equation (10) can be written as

$$y_F = \bar{y} + \frac{k_G{}^2}{\bar{y}}. \tag{11}$$

Example. The flat plate in Fig. 14 is immersed in water.

$$F = w\bar{h}A = 62.4(10) \sin 60°(32) = 17{,}300 \text{ pounds};$$

$$y_F = \bar{y} + \frac{I_G}{A\bar{y}} = 10 + \frac{4(8)^3}{12(32)10} = 10.53 \text{ feet.}$$

The total force of 17,300 pounds passes through a point 10.53 feet below D measured down the plane.

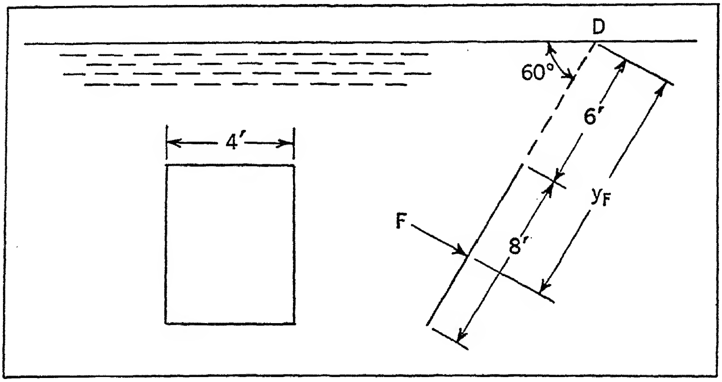


FIG. 14. Plate in water.

15. Total forces on submerged curved surfaces

In some cases the calculation of the total forces acting on a submerged curved surface becomes involved. Frequently such a calculation can be expedited by dealing with components of a total force, rather than with

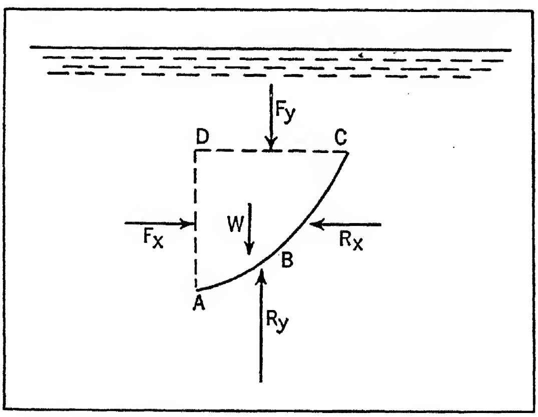


FIG. 15. Curved surface submerged in a liquid.

the total force itself. As an example, refer to the curved surface ABC in Fig. 15. The area DC is the projection of ABC on a horizontal plane, whereas the area AD is the projection of ABC on a vertical plane.

A study can be made of the forces acting *on* the body of fluid $ABCD$. The forces F_x and F_y can be computed by the methods previously outlined.

W is the weight of the fluid in the volume $ABCD$, and acts through the centroid of this volume. The total force of the surface ABC *on* the body of fluid is represented by the combination of the vertical force R_y and the horizontal force R_x. For equilibrium the sum of all the forces in any direction must equal zero, and the sum of all the moments about any point must equal zero. In the horizontal direction, then, $R_x = F_x$. The line of action of R_x coincides with that of F_x. This condition explains the rule that is sometimes stated, that the total horizontal component of the forces acting on any area is equal to the horizontal force acting on a plane which is the vertical projection of the curved area. In the vertical direction, $F_y + W - R_y = 0$.

16. Buoyancy

When a body is partly or completely immersed in a static fluid, as shown in Fig. 16, every part of the surface in contact with the fluid is pressed on by the latter, the pressure being greater on the parts more deeply immersed. The resultant of all these forces exerted by fluid pressure is an upward, buoyant, or lift force. Application of a convenient concept will show the magnitude and direction of this force for any body form.

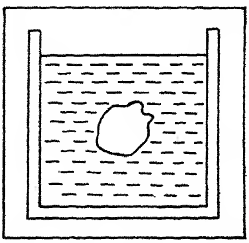


FIG. 16. Body in a fluid.

The pressure on each part of the surface of the body is independent of the body material. Imagine the body, or as much of it as is immersed, to be replaced by fluid like the surrounding mass. This substituted fluid will experience the pressures that acted on the immersed body, and this fluid will be at rest. Hence, the resultant upward force on the substituted fluid will equal its weight, and will act vertically through its center of gravity.

Therefore, following Archimedes' principle, a body immersed in a fluid experiences a buoyant, or lift force equal to the weight of the displaced fluid. This static lift force acts vertically through the center of gravity of the displaced volume, which point is usually called the center of buoyancy.

17. Stability

A study of stability is necessary in many cases involving a body in a fluid. A body is said to be *stable* or in *stable equilibrium* when it will return to its original position after being slightly displaced. A pendulum, pivoted above the center of gravity of its mass, swings back to its original position after being slightly displaced. There is a restoring force; the pendulum is in stable equilibrium.

A body is in *neutral equilibrium* if it remains in the position attained after being slightly displaced. A block resting on a flat horizontal surface is in neutral equilibrium.

A body is *unstable*, or in *unstable equilibrium*, if it continues moving in the direction of displaced motion. A pencil balanced on its point is unstable.

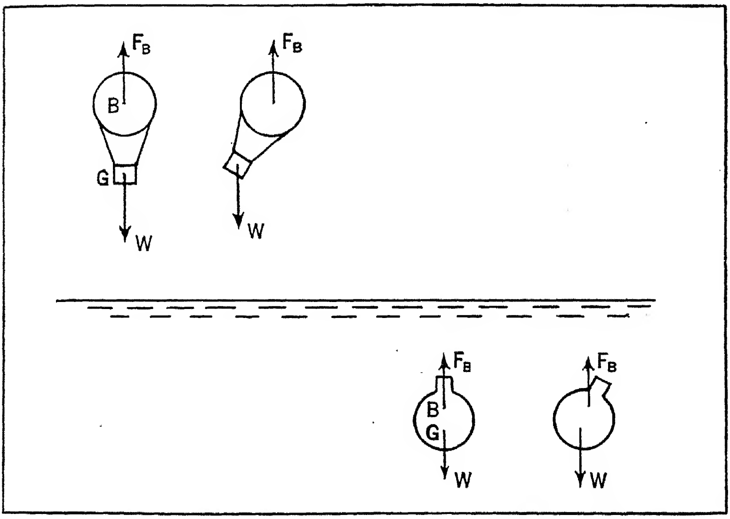


FIG. 17. Equilibrium and displaced positions of bodies immersed in a fluid.

In Fig. 17, B is the center of buoyancy, and G is the center of gravity of the body. Static equilibrium is reached when the buoyant force F_B equals the weight W and when the forces are in a vertical line. The equilibrium of a completely immersed body is stable, as for the balloon and the submarine, when the center of buoyancy is *above* the center of gravity. If such a body is tilted, a couple is developed which tends to swing the body back to its original position.

Figure 18*a* indicates a floating body, like a surface vessel. The buoyant force F_B equals the weight W. For equilibrium it is necessary that G and B lie on a vertical line. G is usually above B for surface ships except some fast sailing boats. In Fig. 18*b* the line of action of the buoyant force intersects the center line of the body at point M. Point M is called the *metacenter*. For the case shown in Fig. 18*b* there is a couple tending to rotate the body back to the original position. The magnitude of this couple equals the product of the weight and the perpendicular distance between the line of action of the weight and the line of

action of the buoyant force. For a floating body to be stable, *it is necessary that the metacenter lie above the center of gravity.*

The hydrometer, used for determining the specific gravity of liquids, is an important physical application of the floating of partly immersed bodies.

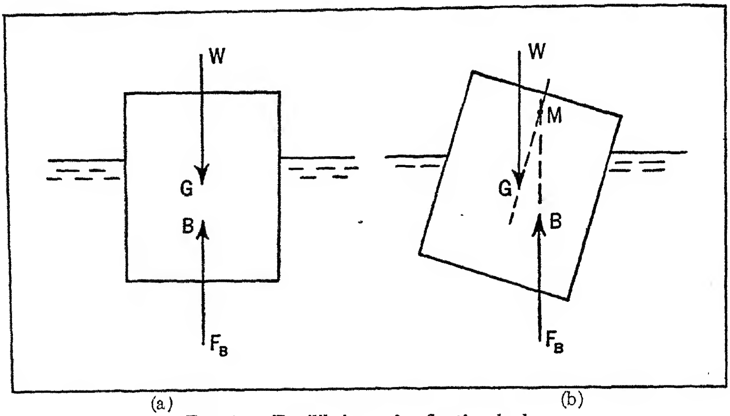


FIG. 18. Equilibrium of a floating body.

18. Pressure-height relations for compressible fluids

The fundamental equation of fluid statics was integrated for the case of incompressible fluids. There remains the case for compressible fluids. As one example, consider an isothermal layer of gas. Then the simple equation of state for gases shows that

$$pv = \text{constant} = p_1 v_1 = RT_1 = \frac{p}{w}.$$

Substitution in Equation (3) gives

$$\int_1^2 \frac{dp}{w} = -\int_1^2 dz, \qquad RT_1 \int_1^2 \frac{dp}{p} = -\int_1^2 dz,$$

$$z_2 - z_1 = RT_1 \log_e \frac{p_1}{p_2}. \tag{12}$$

Equation (12) is sometimes called the *barometric-height* relation. For an isothermal atmosphere a measurement of the temperature and the pressure (as by a barometer) at two different levels will provide data for the calculation of the height difference. Actually, there usually is a decrease in temperature with increasing height in the atmosphere. For dry air R is 53.3 feet per degree Fahrenheit. For medium damp air R is about 53.6 feet per degree Fahrenheit.

For the more general case, the polytropic,

$$pv^n = \text{constant}.$$

Using the relations

$$pv^n = p_1 v_1^n, \qquad v = \frac{1}{w}, \qquad \frac{p_1}{w_1} = RT_1,$$

substituting in the fundamental Equation (3), and integrating, gives the pressure-height relation:

$$z_2 - z_1 = \frac{n}{n-1} RT_1 \left[1 - \left(\frac{p_2}{p_1} \right)^{\frac{n-1}{n}} \right]. \tag{13}$$

For a dry adiabatic atmosphere $n = k = 1.4$. For a moist atmosphere n is lower than 1.4, an average approximate value being $n = 1.2$.

This article and Article 10 illustrate how the pressure-height relation for *any* fluid can be obtained. The functional relation between p and w is inserted in the fundamental equation of statics, and the integration is carried out. Applications for the case of incompressible fluid have been given in the previous articles. Other applications for compressible fluids will be shown in Article 20.

19. Standard atmosphere

The atmosphere, that body of fluid in which we live, is an envelope of gas surrounding the earth. A *standard atmosphere,* as shown in Fig. 19, has been officially adopted for certain uses in the United States. This *standard atmosphere* is an assumed, arbitrary condition of the atmosphere, which is necessary for evaluating the performance of aircraft and calibrating air instruments.

For example, the conventional altimeter (ăl-tĭm′*e*-ter) is simply a modified aneroid barometer having a dial graduated to show altitude instead of pressure. Generally, the instrument indicates the height above sea level. It is calibrated according to the altitude-pressure relation of the standard atmosphere. Care must be exercised in using and interpreting a conventional altimeter. Such an instrument may under-read in summer and may over-read in winter. Temperatures and pressures existing in a flight may differ from those employed in the calibration, hence give an inaccurate altitude reading.

Standard air at sea level is air at 59° Fahrenheit (15° centigrade) under a pressure of 29.92 inches (760 millimeters) of mercury (14.7 pounds per square inch). For these conditions the air has the following properties:

specific weight w = 0.07651 pounds per cubic foot,
density ρ = 0.002378 slugs per cubic foot.

Unless otherwise specified, problems dealing with air in this book will use standard air.

Besides the units already mentioned, the unit *bar* is sometimes used for pressure in sound and weather work. For example, the United States Weather Bureau has adopted the bar unit for reporting pressure on daily weather maps. The following conversions apply:

$$1 \text{ millibar} = 1000 \text{ dynes per square centimeter},$$
$$1000 \text{ millibars} = 29.53 \text{ inches of mercury}.$$

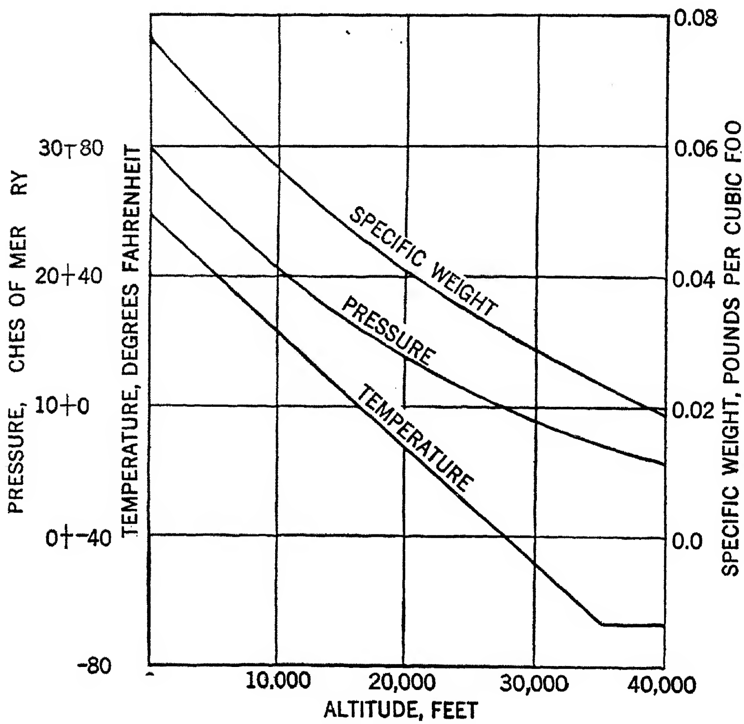


FIG. 19. Standard atmosphere.[2]

1013.2 millibars corresponds to the "standard" pressure of 29.92 inches of mercury.

In the lower layer of the atmosphere, or *troposphere,* the temperature usually decreases with height. In the upper layer, or *stratosphere,* the temperature is practically constant. The *tropopause* is the zone between the troposphere and the stratosphere; the tropopause is the gradual transition between these two. The height to the base of the stratosphere

[2] Data from *Standard Atmosphere Chart,* Miscellaneous Publication of the Bureau of Standards, No. 82.

varies with latitude and season; on the average, the height is about 35,000 feet.

20. Convective stability of masses in fluids

Some important features of stability, other than those previously treated, can be conveniently illustrated by reference to the vertical movement of air masses (convection) in the atmosphere.

If an air mass at the surface were moved aloft (by some thermal or mechanical means, as by a gust of wind or a mountain) the air mass would cool as it expanded. Such cooling is a common occurrence, and may be sufficient to cause condensation (in moist air) with the subsequent formation of rain, fog, or cloud. If the air mass moved earthward from aloft, the compression would result in a temperature rise, in a manner similar to the familiar action in a tire pump.

It is commonly considered that such vertical movements in the atmosphere follow the adiabatic law; the mass motion is so rapid that the heat addition or removal is negligible. It is usually assumed that when no condensation occurs, the process follows the dry adiabatic law; that is, $n = 1.4$ in the relation $pv^n =$ constant. The temperature change per unit height, or *temperature gradient,* can be computed with the aid of relations already established.

Eliminating v from the equations $pv^n =$ constant and $pv = RT$ gives

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}}.$$

Substitution of the foregoing in Equation (13) yields

$$z_2 - z_1 = \frac{n}{n-1} RT_1 \left[1 - \frac{T_2}{T_1}\right]$$

$$\frac{-T_1}{-z_1} = -\left(\frac{n-1}{n}\right)\frac{1}{R}. \tag{14}$$

For dry air following the adiabatic process $n = 1.4$ and $R = 53.3$ feet per degree Fahrenheit. With these units the temperature gradient is

$$\frac{T_2 - T_1}{z_2 - z_1} = -0.0054 \text{ (nearly)}.$$

Imagine a mass of air at the surface, as represented by point A in Fig. 20. If this air mass were to move aloft it would experience a drop in temperature of about 5.4° Fahrenheit for each 1000 feet, as indicated by the dotted line marked "dry adiabatic." The line labeled "unstable" indicates an existing condition of the surrounding atmosphere (as might be determined by a balloon or airplane sounding). At a certain level, say 3000 feet, the surrounding atmosphere is *colder* than the air mass which

has just moved to this altitude. A lower temperature means denser air; the mass moved from the surface would be pushed upward further by the buoyant force; the condition is *unstable.*

The line in Fig. 20 labeled "stable" represents a temperature gradient less than the dry adiabatic. The stability can be reasoned out in a manner similar to that of the previous case. If the observed temperature gradient in the atmosphere were equal to the adiabatic, the equilibrium would be neutral. The temperature of the moved air mass would be the same as the surrounding air; there would be no force pushing the air mass up or down.

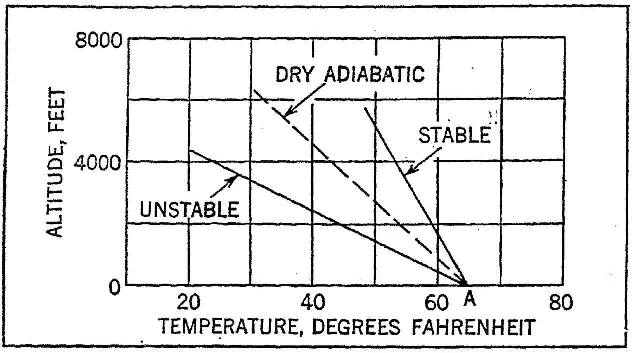


Fig. 20. Types of equilibrium in the atmosphere.

Cumulus clouds, characterized by a horizontal flat base and a fluffy cauliflower top, may be formed by a vertical movement of moist air in an unstable atmosphere, with condensation starting at the base. If a mild shower results, it is sometimes called an *instability shower.* A violent instability shower may develop into a thunderstorm, a condition accompanied by thunder, lightning, and sometimes hail. Vertical observations in the atmosphere, together with studies of stability, may be very helpful in the forecasting of thunderstorms. Questions of convective stability are also involved in applications dealing with heat-transfer equipment.

21. Accelerated liquids in relative equilibrium

If a container with a fluid in it is given a constant or uniform acceleration, the fluid finally comes to rest relative to the container; the condition is one of relative equilibrium. The body of the fluid is free from shear forces; it is in static equilibrium.

Let m be the mass of a fluid element, and a its linear acceleration (given in magnitude and direction). According to Newton's law, the resultant of all the forces acting on the element, including those due to the pressure of the surrounding fluid, must equal ma. Let F be the

resultant of all the impressed forces. Then $F = ma$. D'Alembert's principle states that this equation can be written as $F - ma = 0$. The fictitious force $-ma$ is sometimes called the reversed effective force or the inertia force. D'Alembert's principle thus makes it possible to reduce a problem in dynamics to an equivalent problem in statics by the introduction of an inertia force (or inertia forces).

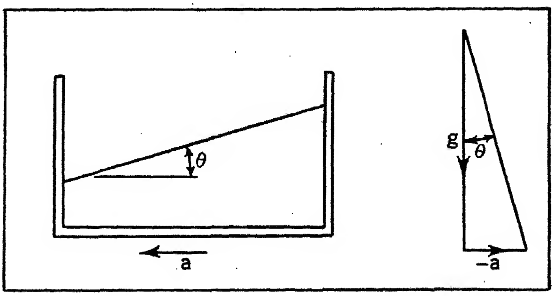


FIG. 21. Liquid in tank with constant linear acceleration.

One example is illustrated in Fig. 21. A tank containing a liquid has a constant horizontal linear acceleration a. The pressure distribution will be the same as if the liquid were at rest under gravitational acceleration and an acceleration $-a$. The effect is the same as if the direction of gravity were turned an angle θ back from the vertical, such that

$$\tan \theta = \frac{a}{}. \tag{15}$$

The free surface is a plane tilted through an angle θ. The pressure at any point on the bottom of the container is found by the relation $\Delta p = wh$, where h is the height to the free surface at the point in question.

2. Relative equilibrium of rotating liquids

Another example of the principles stated in Article 21 is found in the ction of a liquid in a container rotating about a vertical axis with con- nt angular velocity ω (Greek letter omega). ω is expressed in radians r unit time. Relative equilibrium is reached a short time after start- g; the liquid then rotates as a solid body. Such a motion is sometimes lled a *forced vortex*. The free surface of the liquid is curved, as indi- ted in Fig. 22.

There are three forces acting on the fluid element at point A in Fig. 22. e force is the weight of the element W. Each element experiences an celeration towards the axis of rotation; this centripetal acceleration $\omega^2 x$. The inertia force $(W/g)\omega^2 x$ acts in a radial direction away from e axis of rotation. The force P is the resultant force due to the pressure

of the surrounding fluid particles. Because there is no relative sliding between the particles, P is normal to the curved surface. Since these three forces are in equilibrium,

$$P \sin \theta = \frac{W}{g}\,\omega^2 x, \qquad P \cos \theta = W, = mg$$

$$\tan \theta = \frac{\omega^2 x}{g} = \frac{dy}{dx}. \tag{16}$$

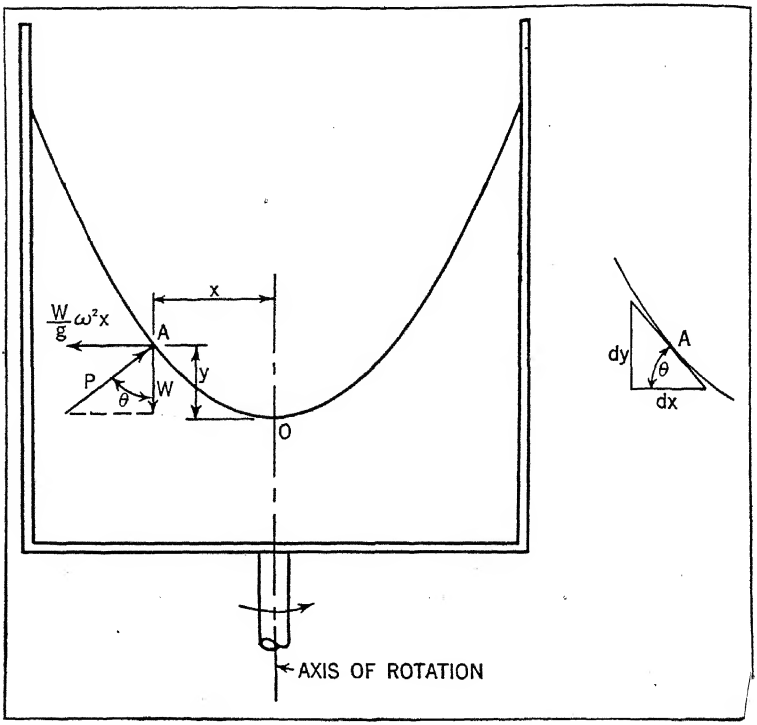


Fig. 22. Rotating body of liquid.

Integration gives the equation of the curved free surface,

$$y = \frac{\omega^2 x^2}{2g}, \tag{17}$$

which is the equation of a parabola. The pressure at any point on th bottom of the container is found from the relation $\Delta p = wh$, where h i the height to the free surface above that point.

An example of a forced vortex is found in the rotation of liquid in th impeller of an idealized centrifugal pump with no flow through the pum

If fluids of different densities are placed in a stationary vessel, static equilibrium is reached when the density in any horizontal layer is constant, and the less dense fluid is above the more dense fluid. There are important engineering applications involving the settling and separation of fluids and particles, for example oil and water or oil and foreign particles. In a stationary vessel, with only gravity force acting, the settling may take place slowly. By rotating the vessel at a high speed, however, an acceleration much greater than the gravitational acceleration can be obtained, and a much quicker separation effected.

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The Physics of Solids and Fluids by P. P. Ewald, T. Pöschl, and L. Prandtl. Blackie and Son, Ltd., London, 1932, page 153.

Statics by H. Lamb. Cambridge University Press, 1933.

Physics of the Air by W. J. Humphreys. McGraw-Hill, New York, 1929.

PROBLEMS

8. Find the pressure in tons per square yard at a depth of 15 fathoms in the sea (specific gravity = 1.025; 1 fathom = 6 feet).

9. How many feet of water are equivalent to 25 pounds per square inch? How many inches of mercury are equivalent to 25 pounds per square inch?

10. A tube is closed at one end and open at the other. This tube, originally containing air at standard conditions, is lowered vertically into the sea (specific gravity = 1.025), with the closed end up. When the tube was raised, it was found that the water had risen 0.80 of the length of the tube. Find the depth reached.

11. Two cylindrical diving bells of the same size are held immersed in water. Each is closed at the top and open at the bottom. The water inside bell A stands 4 feet below the free surface outside, while the water in bell B stands 6.5 feet below the free surface outside. If the interiors are connected, what is the new water level for the same weight of atmospheric air?

12. A cylindrical diving bell 10 feet high, originally full of air, is lowered until the fresh water rises 4 feet in the interior. The cross-sectional area of the bell is 10 square feet. What is the depth of the top of the bell below the surface? How many cubic feet of air at atmospheric pressure must be pumped in, in order that the water may be expelled from the interior?

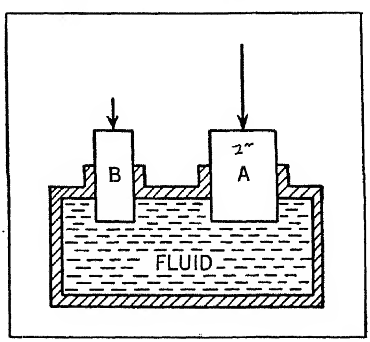

FIG. 23.

13. Piston A in the fluid press shown in Fig. 23 has a diameter of 2 inches. A force of 20 pounds at piston A exerts

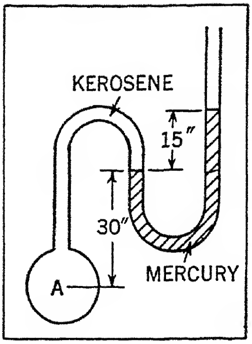

Fig. 24.

a force of 1000 pounds at piston B. Neglecting friction in the piston guides, what is the diameter of piston B?

14. A vertical glass U-tube is employed to measure the pressure of air in a pipe. The water in the arm of the tube open to the atmosphere stands 7.5 inches higher than that in the arm connected to the pipe. Barometric pressure is 30.2 inches of mercury, and the temperature is 60° Fahrenheit. What is the absolute pressure in the pipe, and the density of the air?

15. For the arrangement shown in Fig. 24, determine the gage pressure at A. The kerosene has a specific gravity of 0.82.

16. For the inclined-tube draft gage shown in Fig. 25, compute the gage pressure at B if the right leg is open to the atmosphere. The oil has a specific gravity of 0.87.

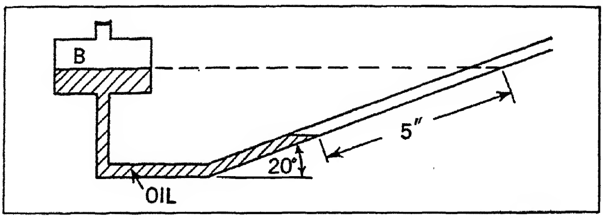

Fig. 25.

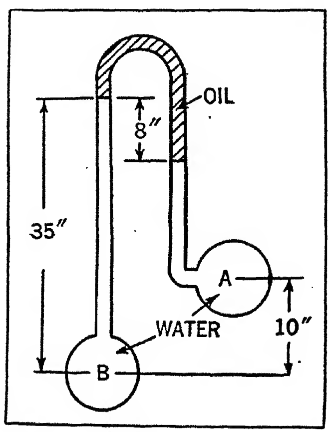

Fig. 26.

17. For the arrangement in Fig. 26, calculate the pressure difference between points A and B. Specific gravity of the oil is 0.85.

18. For the arrangement shown in Fig. 27, calculate the pressure difference between A and B.

19. A rectangular tank 4 feet long, 3 feet wide, and 5 feet deep is filled with glycerine. What is the total force on the bottom, on one side, and on one end? What is the location of the total force on one end?

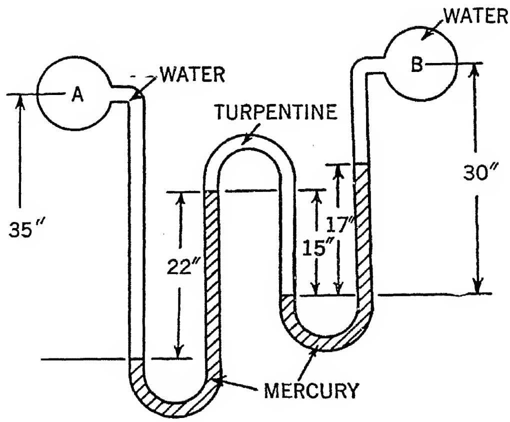

FIG. 27.

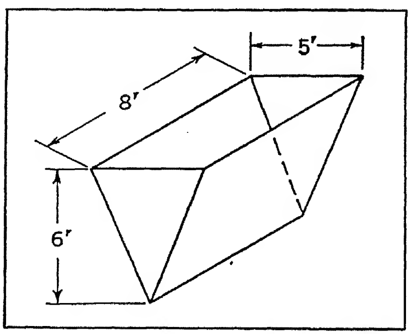

FIG. 28.

20. The triangular tank in Fig. 28 is filled with turpentine. What is the total force on one end and one side? What is the location of the force on one end?

21. A cylindrical tank 3 feet in diameter has its axis horizontal. At the middle of the tank, on top, is a pipe 2 inches in diameter, which extends vertically upwards. The tank and pipe are filled with castor oil, with the free surface in the 2-inch pipe at a level 10 feet above the tank top. What is the total force on one end of the tank?

22. A cylindrical container 10 inches in diameter, with axis horizontal, is filled with mercury up to the cylinder center. Find the total force acting on one end.

23. A circular disk 8 feet in diameter is in a plane sloping 30° from the vertical. A lye solution (specific gravity = 1.10) stands above the disk center to a depth of 10 feet. Calculate the magnitude, direction, and location of the total force of the solution on the disk.

24. A rectangular gate 7 feet high and 5 feet wide is placed vertically on the side of an open rectangular container of oil (specific gravity = 0.84). The free oil surface is 4 feet above the upper edge of the gate. What force must be applied at the upper edge of the gate to keep it closed if the gate is hinged at the lower edge?

25. A tank, separated in the center by a vertical partition, contains water on one side to a depth of 10 feet. The other side contains nitric acid (specific gravity = 1.50) to a depth of 12 feet. A rectangular opening in the center partition, 1.5 feet wide by 2 feet high, is closed by a flat plate. The plate is hinged at its upper edge, which is 2 feet above the bottom of the tank. What force applied at the lower edge of the plate is necessary to keep it closed?

26. A rock weighs 90 pounds in air and 62 pounds in water. Find its volume and specific gravity.

27. A barge 60 feet × 30 feet sank $1\frac{1}{4}$ inches in fresh water when an elephant was taken aboard and located centrally. What was the elephant's weight?

28. A rectangular steel box, 40 feet × 20 feet × 25 feet high, and weighing 80 tons, is to be sunk in water. How deep will it sink when launched? If the water is 18 feet deep, what weight must be added to cause it to sink to the bottom?

29. An iceberg has a specific weight of 57.2 pounds per cubic foot. What portion of its total volume will extend above the surface if it is in fresh water?

30. An airship contains 5 tons of hydrogen at the surface of the earth. What will be the lift in air, and the volume of the hydrogen at 56° Fahrenheit and 30.2 inches of mercury? Assume that the air and hydrogen are at the same temperature.

31. An airship is to have a lift of 40 tons when the gas temperature equals that of the air. What weight of helium must it have in its cells?

32. The hull of a certain ship has sides sloping outward and upward at the water line. Explain why this design is more stable than a hull with parallel sides.

33. For an assumed isothermal atmosphere, show that the pressure becomes zero at an infinite height.

34. The barometric pressure at sea level is 30.10 inches of mercury when that on a mountain top is 29.00 inches. If the air temperature is constant, at 58° Fahrenheit, what is the elevation at the mountain top?

35. Atmospheric pressure at the ground is 14.70 pounds per square inch at a temperature of 60° Fahrenheit. Determine the absolute pressure 12,000 feet above the ground: (a) if the air is incompressible, (b) if the air follows the isothermal relation, and (c) if the air follows the dry adiabatic relation.

36. An air mass at 70° Fahrenheit is quickly forced up vertically a distance of 8000 feet. At this altitude the surrounding atmosphere has a temperature of 20° Fahrenheit. At this level, will the air mass be pushed up or down?

37. A rectangular tank 4 feet wide, 9 feet long, and 5 feet deep contains water to a depth of 3 feet. It is accelerated horizontally at 8 feet per second per second in the direction of its length. Calculate the depth of water at each end of the tank, and the total force on each end of the tank.

38. A cylindrical tank 1 foot in diameter and 3 feet high is half full of glycerine. The tank is rotated about its vertical axis. What speed of rotation will cause the liquid to reach the top? What will then be the maximum pressure?

39. A simple centrifugal oil separator consists essentially of a cylindrical container rotating about its axis. In some cases the impurities inherent in an oil combine both chemically and mechanically with foreign impurities to form what is commonly called sludge. If the sludge is denser than the oil, at what points of the simple separator should pipes be located for removing the clean oil, and for removing the sludge?

CHAPTER 3

Kinematics of Fluid Flow

> All that we see distinctly in the motion of a body is that the body traverses a certain distance and that it takes a certain time to traverse that distance. It is from this one idea that all the principles of mechanics should be drawn, if we wish to demonstrate them in a clear and accurate way.—D'ALEMBERT.[1]

A study of fluids in motion requires considerations of fluid properties, kinematics, and energy and force relations. The present chapter covers some features of kinematics, or the geometry of motion, without regard to the forces causing that motion. The next chapter discusses energy relations, and subsequent chapters will treat in detail the forces involved in fluid flow.

23. Path lines and streamlines

Certain lines are helpful in describing fluid motion. A *path line* is a line made by a single particle as it moves during a period of time. The trace made by a single smoke particle, as it issues from a chimney, would be a path line. A path line may be obtained with a long exposure on a fixed photographic plate.

A *streamline* is a line which gives the velocity direction of the fluid at each point along the line. Consider a series of particles at an instant in a flowing fluid. Imagine that at each particle a line is drawn showing the direction of the instantaneous velocity at each point. A smooth curve tangent to each of these lines would be a streamline. Note that a path line refers to the path of a *single particle*, whereas a streamline refers to an instantaneous picture of the velocity direction of a *number* of particles. Streamlines are analogous to the lines of force in a force field, like a magnetic field.

24. Steady flow—systems of reference

Steady flow is defined as a flow in which the velocity, pressure, density, and other such characteristics at a point do not change with time. Steady flow specifies a limitation on *time* variation, not space variation. In steady flow, for example, the velocity at one point in a body of fluid may differ from that at another point. The flow is unsteady if the

[1] *A Source Book in Physics* by W. F. Magie. McGraw-Hill, New York, 1935.

velocity and other such characteristics vary with time at a point. Path lines are identical with streamlines in steady flow.

Sometimes it is convenient to change a nonsteady flow into a steady flow by shifting the reference system or the position of an observer. Picture the flow around a bridge pier. An observer on the pier will note steady flow if the streamlines do not change with time. On the other hand, an observer in a rowboat moving with the same stream will find the flow unsteady. If an observer on an airship (or airplane) witnesses steady flow, another observer on the ground would find an unsteady flow of the fluid as the aircraft approaches and passes.

25. Equation of continuity

The *stream-tube* indicated in Fig. 29 may be formed by the walls of a pipe, by some other piece of equipment, or by a surface of streamlines. No fluid crosses the walls of the stream-tube. The *equation of continuity* for steady flow in this stream-tube is a special case of the general physical law of the conservation of matter. This equation states that the weight (or mass) of fluid passing any section per unit time is constant. Let V represent the average velocity at any section, and A the area at this section. Let the subscript 1 refer to a particular section, and the subscript 2 refer to another section. Then:

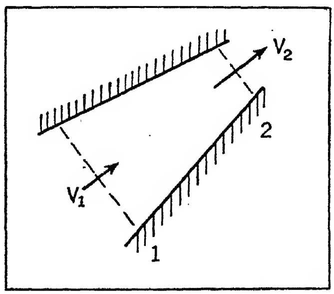

FIG. 29. Stream-tube.

weight rate of flow = area × velocity × specific weight = constant.

$$A_1V_1w_1 = A_2V_2w_2. \tag{18}$$

For liquids and gases which can be treated as incompressible, w is constant, and Equation (18) takes the special form

$$Q = A_1V_1 = A_2V_2 = \text{constant.} \tag{19}$$

The product $Q = AV$ is commonly called *rate of discharge* or *volume rate of flow*, and may be expressed in cubic feet per second.

Streamlines can be made visible in various ways, by using smoke, aluminum powder, or lycopodium powder. Figures 30 and 31 show actual smoke-tunnel photographs taken in the Purdue Fluid Mechanics Laboratory. The air flow, between two glass plates $\frac{1}{4}$ inch apart, is from left to right. Titanium tetrachloride smoke was introduced at equidistant points before the model to indicate the streamlines. For this case, and if it is assumed that the fluid is incompressible, the continuity equation shows that a converging of the streamlines is associated with an

increase in the velocity of the fluid between the streamlines. There is a crowding of streamlines above the tapered section and in the nozzle throat. At the nozzle throat, particularly, the velocity is considerably higher than that some distance upstream.

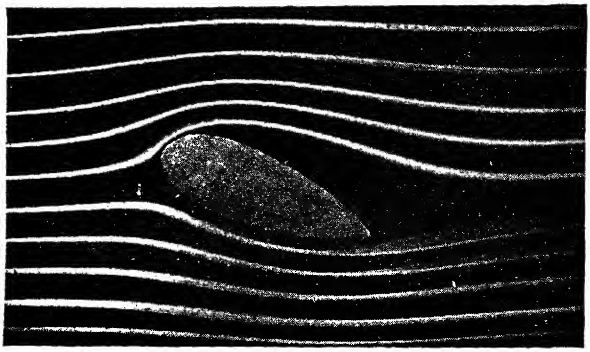

FIG. 30. Flow around a tapered section.

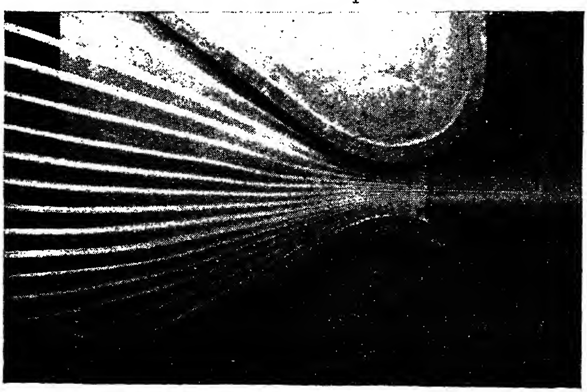

FIG. 31. Flow through a nozzle.

26. Velocity distribution

A general problem is to map out the velocity distribution in the field of a flowing fluid. If the velocity at each point in the fluid is known, an application of an energy equation[2] yields the pressure at each point. The total force acting on a body in a fluid can be determined from the pressure distribution around the body.

A comprehensive treatment of velocity distribution will not be given in this chapter. An elementary, simplified treatment will be presented, however, in order to illustrate some of the fundamentals involved and a

[2] See Chapter 4.

general method of approach. The following assumptions will be made for the remainder of this chapter: (1) a frictionless fluid, without eddies; (2) two-dimensional flow, that is, flow which is identical in parallel planes; and (3) steady flow.

Certain simple types of flow will be combined to give various resultant streamline patterns. In such a *potential* or *possible* flow the streamlines may be considered as solid boundaries, since no fluid *crosses* a streamline. A closed streamline, then, may be replaced by a solid body without affecting the flow.

27. Two-dimensional source and sink flow

In Fig. 32 fluid enters the opening and moves radially outward. Strictly speaking, the opening should be infinitesimal; at the center the

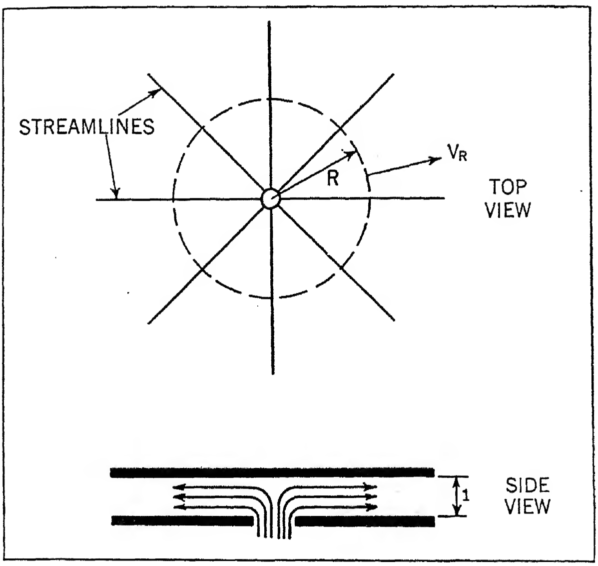

FIG. 32. Two-dimensional frictionless source flow.

velocity approaches infinity. These singular conditions will be avoided by confining attention to a radius some distance away from the center. The flow shown in Fig. 32 will be referred to as one due to a *source*. Flow in the opposite direction is a *sink*. In either case the streamlines are radial. Let Q = the rate of fluid discharge, and V_R the radial velocity at radius R. Then, from Fig. 32,

$$Q = 2\pi R V_R,$$

$$V_R = \frac{Q}{2\pi R}. \tag{20}$$

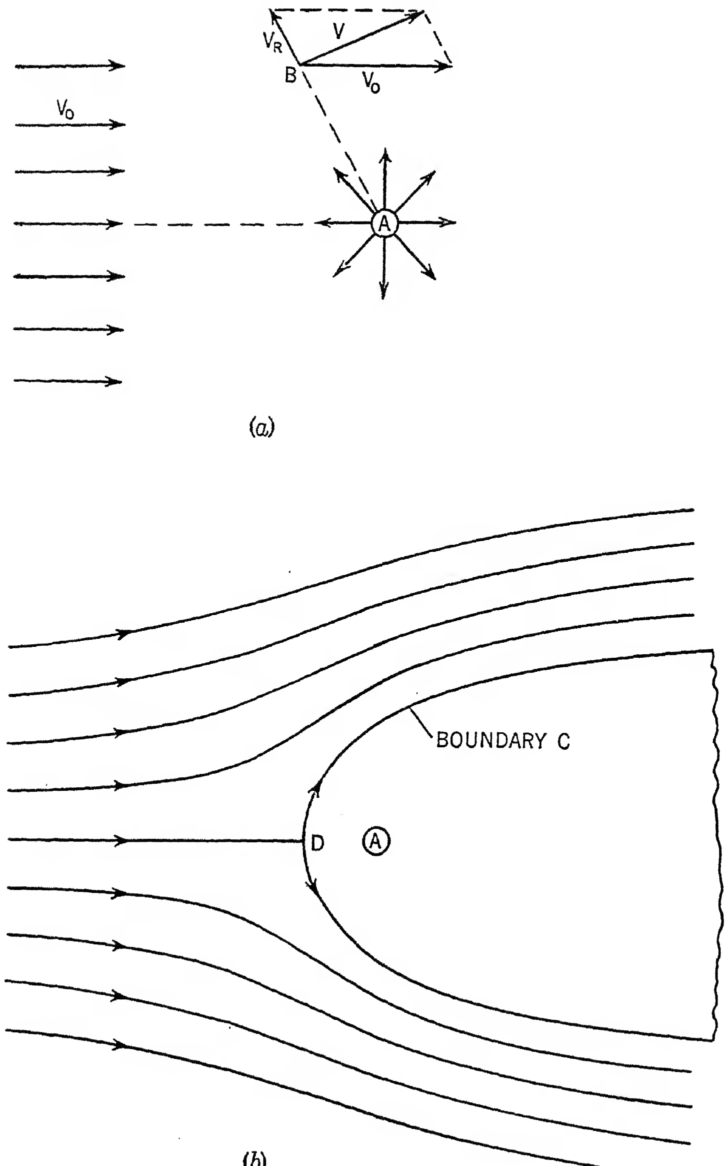

FIG. 33. Two dimensional frictionless flow around a half-streamline body.

28. Superposition of rectilinear, source, and sink flows

Fig. 33a shows a uniform rectilinear flow with a velocity V_0; the streamlines are parallel. A source is placed at point A. At point B the velocity component V_0 is added vectorially to the radial velocity V_R (due to the source) to give the resultant velocity vector V. The streamline at B is tangent to the velocity vector V. Extension of this vector addition to other points in the field gives a series of streamlines as shown in Fig. 33b. A solid body can be substituted for any of the streamlines because no fluid crosses a streamline. In Fig. 33b the boundary C has

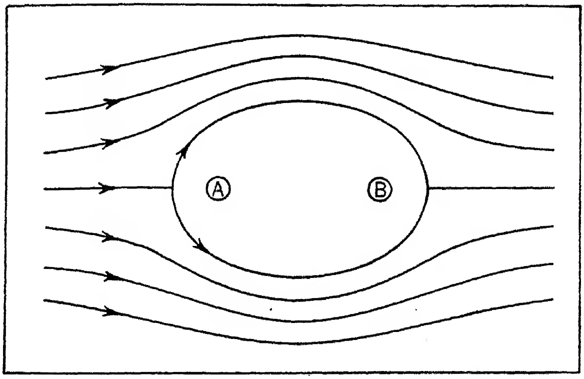

FIG. 34. Two-dimensional frictionless flow around an oval.

been selected as a body. This superposition of a rectilinear flow on a source flow gives a resultant flow which is the flow around the *half-streamline* body C. Various shapes of bodies can be obtained by using different values of the velocity V_0 and the source strength Q.

The flow around the oval in Fig. 34 is obtained by combining a rectilinear flow V_0 (along the line AB) with a source at A and a sink at B. The discharge rate Q from the source exactly equals the input rate to the sink. Different oval shapes can be obtained by employing different values of Q, V_0, and distances AB. If the point A is brought closer to B, the oval approaches a circle as its limit, like that shown in Fig. 35. This limiting combination of a source and sink is called a *doublet*.

Various graphic techniques can be developed for expediting the vector addition, for plotting streamlines, and for obtaining a qualitative picture of the flow. For computation purposes involving a large class of fluid motions, the velocity distribution can be obtained analytically by means of *potential functions*. Potential functions are helpful in other studies besides those of fluid flow. A simplified treatment of the use of these functions, in mathematically superposing simple flows, will be given in Chapter 19.

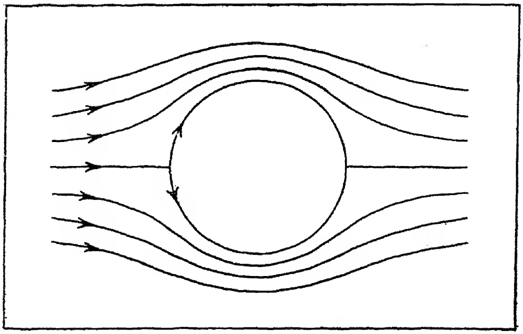

FIG. 35. Two-dimensional frictionless flow around a cylinder.

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An Introduction to Fluid Mechanics by A. H. Jameson. Longmans, Green and Co., New York, 1937.

PROBLEMS

40. A pipe 12 inches in diameter reduces to a diameter of 6 inches, and then expands to a diameter of 10 inches. If the average velocity in the 6-inch pipe is 15 feet per second, what is the average velocity at the other sections for *any* incompressible fluid?

41. A pipe 8 inches in diameter expands to 14 inches diameter. Air flows through the pipe at a rate of 15 pounds per minute. At one section, in the 8-inch pipe, the gage pressure is 40 pounds per square inch, and the temperature is 120° Fahrenheit. At another section, in the 14-inch pipe, the gage pressure is 25 pounds per square inch and the temperature is 80° Fahrenheit. Barometric pressure is 30.0 inches of mercury. What is the mean velocity at each section?

42. For a half-streamline body, as illustrated in Fig. 33, $V_0 = 20$ feet per second, and the source discharge is 80 cubic feet per second. The distance between parallel planes is 1 foot. What is the distance from the center of the source A to the stagnation point D?

CHAPTER 4

Energy Equation for the Steady Flow of Any Fluid

> I was struck with the considerable degree of heat which a brass gun acquires . . . in being bored; . . . the source of the heat generated by friction, in these experiments, appeared evidently to be inexhaustible.
>
> . . . any thing which any insulated body, or system of bodies, can continue to furnish without limitation, cannot possibly be a material substance; and it appears to me to be extremely difficult, if not quite impossible, to form any distinct idea of anything, capable of being excited and communicated, in the manner the heat was excited and communicated in these experiments, except it be MOTION.—RUMFORD.[1]

The energy equation for steady flow is a powerful, easily applied, and useful tool for the engineer. It is simply an energy accounting. The purpose of the following is to discuss this equation, and to indicate its generality and usefulness.

29. Nature of flow

Some features of the type of flow under consideration in this chapter will be discussed first. The fluid may pass through any kind of apparatus. The flow may be through a piece of pipe, a nozzle, a pump, a turbine, some other piece of plant equipment, the passage between streamlines, or a complete plant. Any kind of fluid may be flowing; there is no restriction.

The flow is such that at the entrance section and at the exit section each characteristic can be described by a definite numerical value. If the pressure or temperature at a section fluctuates too much, it may not be possible to assign definite values. Some sort of average is necessary if variations occur. For example, if the velocity varies over a section, it is necessary to decide on some mean value of velocity.

The weight rate of flow must have a definite value. Consider a reciprocating engine or compressor, which takes a quantity of fluid, holds it for a portion of a stroke, and then discharges it. This flow is not steady. The fluctuations may be reduced, however, by including sufficient piping or receiver volume between entrance and exit of the apparatus. Reduction of the fluctuations results in an approach to steady flow. The actual flow in many pieces of engineering equipment closely approximates steady flow.

[1] *A Source Book in Physics* by W. F. Magie. McGraw-Hill, New York, 1935.

30. Work and energy

Work is defined as the product of a force times displacement (or distance), with the force acting in the direction of the displacement. Work may be expressed in such units as foot-pounds. Energy may be defined as the capacity for doing work; energy is expressed in the same units as work.

A body may have several forms of energy. A mass m moving with a velocity V has kinetic energy equal to $\frac{1}{2}mV^2$. A body of weight W ($W = mg$) at a height z above some arbitrary datum has potential energy equal to Wz with respect to the datum. Heat is thermal energy in transition, and may be expressed in British thermal units (B.t.u.); 1 B.t.u. is approximately equal to 778 foot-pounds. The internal or intrinsic energy of a substance is energy stored within the substance, and is due to the activity and spacing of the molecules.

Power is the rate at which work is performed: 1 horsepower equals 550 foot-pounds per second. Power has the dimensions of a force times velocity.

31. Energy equation for the steady flow of any fluid

The general energy equation applies to any change of state of a fluid; it asserts that

The heat added *to* unit weight of the flowing fluid between entrance and exit	the work transferred *to* (done upon) unit weight of the flowing fluid between entrance and exit	the total gain in energy of unit weight of the flowing fluid between entrance and exit.

A particular set of common units will be employed in order to illustrate the use of this energy balance. Any other system of consistent units could be adopted. Figure 36 illustrates the notation.

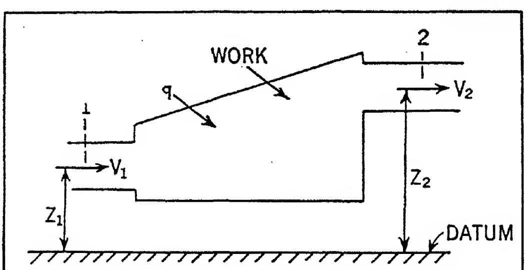

FIG. 36. Notation for steady flow through apparatus.

The foregoing word equation can be expressed mathematically as

$$q + \frac{p_1v_1}{778} - \frac{p_2v_2}{778} + \frac{\text{work}}{778} = u_2 - u_1 + \frac{V_2^2 - V_1^2}{2g(778)} + \frac{z_2 - z_1}{778} \qquad (21)^{2}$$

where q = heat transferred to the fluid, British thermal units (B.t.u.) per pound of fluid flowing; p = pressure, pounds per square foot; v = specific volume, cubic feet per pound; work = mechanical work, foot-pounds per pound of fluid flowing; u = internal energy, B.t.u. per pound of fluid; V = velocity, feet per second; g = 32.2 feet per second per second; z = elevation above some arbitrary datum, feet. Subscript 1 refers to entrance, subscript 2 refers to exit.

Each term of Equation (21) is expressed in B.t.u. per pound of flowing fluid. The total increase in energy consists of the gain in internal energy, gain in kinetic energy, and gain in potential energy; this total increase is expressed by the entire right-hand side of Equation (21). Other forms of energy, such as electrical and chemical, might be included if required by a particular problem. A discussion of the other terms in Equation (21) follows.

The term q is heat, or thermal energy in transition, which passes through the walls of the apparatus, and is, therefore, recognizable from outside the apparatus. Internal frictional effects which *store* thermal energy in the fluid contribute nothing to this term q; such internal energy does not pass through the walls of the apparatus or across the bounding streamlines.

The term $p_1v_1/778$ represents the work transfer or *flow-work* at entrance. It represents the external work (in B.t.u.) done in pushing a pound of fluid across the entrance; it is work transferred *to* the fluid between entrance and exit. The flow-work at exit, $-p_2v_2/778$, represents work transferred *from* the fluid between entrance and exit *to* external bodies outside of the apparatus. The "work" term of Equation (21) includes any other work transferred *to* the fluid by the action of external forces. "Work" is positive for a pump, and negative for a fluid turbine.

The internal energy u for steam, ammonia, and other vapors can be found from suitable tables. For a gas obeying the equation $pv = RT$,

$$u_2 - u_1 = c_v(T_2 - T_1), \qquad (22)^{3}$$

where c_v = specific heat at constant volume.

Some of the terms in Equation (21) are frequently regrouped by using a quantity called *enthalpy* (ĕn-thăl′pĭ):

$$\text{enthalpy} = h = u + \frac{pv}{778}.$$

[2] Sometimes in thermodynamic texts this equation is written as:

$$u_1 + \frac{p_1v_1}{778} + \frac{V_1^2}{2g(778)} + \frac{z_1}{778} + q = u_2 + \frac{p_2v_2}{778} + \frac{V_2^2}{2g(778)} + \frac{z_2}{778} - \frac{\text{work}}{778}$$

There is no essential difference between this equation and Equation (21).

[3] The background for this relation may be found in the usual text on elementary thermodynamics, or in Article 104 in Chapter 12.

Enthalpy is simply a quantity established by arbitrary definition for the sake of convenience. The terms *total heat, heat content,* and *thermal potential* have been used for this quantity, but the designation enthalpy is to be preferred. Using this notation, Equation (21) becomes

$$q + \frac{\text{work}}{778} = h_2 - h_1 + \frac{V_2^2 - V_1^2}{2g(778)} + \frac{z_2 - z_1}{778}. \tag{23}$$

The enthalpy of steam, ammonia, and other vapors can be found from suitable tables. For a gas obeying the relation $pv = RT$,

$$h_2 - h_1 = c_p(T_2 - T_1), \qquad (24)^4$$

where c_p is the specific heat at constant pressure. The use of the specific heat at constant pressure does not imply that the *path* or *process* is necessarily at constant pressure. Equation (24) is simply a means for calculating the change in enthalpy for a gas between states 1 and 2. *Any* kind of process may take place between states 1 and 2.

32. Work done against friction

In considering the flow of real fluids, the question arises as to how friction is taken into account by the general energy equation expressed in the two alternate forms given in Equations (21) and (23). Work done against friction is dissipated as thermal energy at the rubbing point. There are two possibilities:

(1) The thermal energy thus developed may pass through the walls of the apparatus, either wholly or in part. That part of the thermal energy which passes through the walls of the apparatus is included in the q term. If the heat transfer q were measured, frictional heat could not be distinguished from heat from any other source.

(2) That thermal energy due to friction which does *not* pass through the apparatus walls *does not* appear in the q term, but adds to the total energy of the fluid, and hence is included (or concealed) in the right-hand side of Equations (21) or (23).

33. Special applications of the general equation

Frequently, special problems involve simple equations because some of the terms of the general energy equation vanish. If a special, simple form is forgotten or questioned, it is an easy matter to refer to the general equation. It is recommended that the student concentrate on understanding the general energy equation, and then follow the interesting practice of working out the special cases himself as occasion requires. A few illustrations follow.

[4] The background for this relation may be found in the usual text on elementary thermodynamics, or in Article 104 in Chapter 12.

34. Nonflow equation

If there is no flow of fluid, $V_2 = V_1 = 0$, and the flow-work is zero. Taking $z_2 = z_1$, Equation (21) becomes

$$q = u_2 - u_1 - \frac{\text{work}}{778}. \tag{25}$$

35. Flow through ideal nozzle or short tube

For the nozzle or short tube shown in Fig. 37, let $z_2 = z_1$. If the process is adiabatic, $q = 0$. The "work" term is zero. Equation (23) becomes

$$\frac{-V_1^2}{2g(778)} = h_1 - h_2. \tag{26}$$

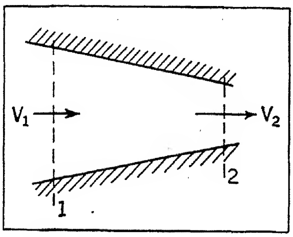

Fig. 37. Nozzle.

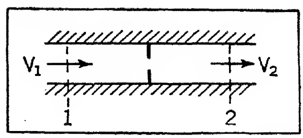

Fig. 38. Orifice.

If V_1 is so small that it may be neglected, then Equation (26) takes the common form

$$V_2 = \sqrt{2g(778)(h_1 - h_2)}. \tag{27}$$

For a gas following the relation $pv = RT$, Equation (27) becomes

$$V_2 = \sqrt{2g(778)c_p(T_1 - T_2)}. \tag{28}$$

36. Throttling processes

Referring to Fig. 38, suppose that the flowing fluid is throttled by some device such as an orifice or a valve. Let $z_2 = z_1$. The "work" term is zero. It is common to assume that the process is adiabatic. If $V_1 = V_2$, Equation (23) becomes

$$h_1 = h_2. \tag{29}$$

Equation (29) does *not* imply a constant-enthalpy *process*. A "constant-something" process means that something remains constant *throughout* the process. Equation (29) only states that the enthalpy at section 2 equals that at section 1. At the orifice plate the fluid velocity is increased (above V_1) to result in a reduction of the enthalpy. Downstream from the plate there occurs turbulent flow in which the increased kinetic energy is dissipated as thermal energy to restore the enthalpy to its initial value.

37. Special cases of flow with small or negligible thermal aspects

For a number of cases it is usually permissible to assume that the process is adiabatic and that the change in internal energy is negligible. For such conditions Equation (21) becomes

$$\frac{p_1v_1}{778} - \frac{p_2v_2}{778} + \frac{\text{work}}{778} = \frac{V_2^2 - }{2g(778)} + \frac{z_2 - z}{778} \tag{30}$$

Each term of Equation (30) is in B.t.u. per pound of fluid. Multiplying each term of Equation (30) by 778 and rearranging, gives

$$\text{work} = \left[\frac{p_2}{w_2} + \frac{V_2^2}{2g} + z_2\right] - \left[\frac{p_1}{w_1} + \frac{V_1^2}{2g} + z_1\right]. \tag{31}$$

Each term of Equation (31) is expressed in foot-pounds per pound of fluid. Any other system of consistent units could be used. Equation (31) is a special application of the general steady-flow energy equation, for the case of an adiabatic process and a negligible change in internal energy.

The term p/w is sometimes called a pressure head and $V^2/(2g)$ a *velocity head.* z might be called an *elevation* or *potential head.* Each of these heads has the net dimension of a length, and may be expressed in such units as feet or inches.

For the special case of a *frictionless incompressible* fluid, with no "work," Equation (31) becomes

$$\frac{p_2}{w} + \frac{V_2^2}{2g} + z_2 = \frac{p_1}{w} + \frac{V_1^2}{2g} + z_1 = \text{constant}. \tag{32}$$

Equation (32) is commonly called Bernoulli's equation. Actually, the expression as written can be ascribed to Bernoulli only by inference; it does not appear in his famous works.

In many real cases the q and $u_2 - u_1$ terms are appreciable. In other real cases the q and $u_2 - u_1$ terms are small. In some of these latter cases the contribution of internal friction effects to the q and $u_2 - u_1$ terms cannot be measured directly, and is either lumped into the "work" term or lumped into a single term. If the work done against friction is included in the "work" term, then the "work" term is the net sum of the following three items:

(a) Positive mechanical work *to* fluid—as in the case of a pump.

(b) Negative mechanical work *to* fluid—as in the case of a fluid turbine.

(c) Negative work *to* fluid—for example, friction.

Frictional dissipation is always a negative quantity of work *to* the fluid.

38. Velocity and pressure distribution

It was pointed out in Chapter 3 that if the velocity at each point in the field of flow is known, an application of an energy equation yields the pressure at each point. Consider the flow around the body shown in Fig. 39, for a frictionless, incompressible fluid with negligible thermal aspects. The pressure is p_0 and the velocity is V_0 in the undisturbed stream to the left. At the *stagnation point* S the fluid velocity is zero,

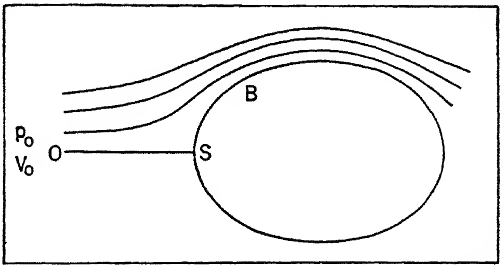

Fig. 39. Flow around body.

and the *stagnation pressure* is p_s. Applying the energy equation to the small stream-tube along the streamline O to S gives

$$\frac{p_0}{w} + \frac{V_0^2}{2g} = \frac{p_s}{w} + 0;$$

since $\rho = w/g$,

$$p_s - p_0 = \frac{1}{2}\rho V_0^2. \tag{33}$$

The term $\rho \frac{V_0^2}{2}$ is commonly known as the *impact* or *dynamic* pressure of the stream. This term occurs frequently in studies of fluid meters, and in studies dealing with the total forces acting on moving submerged bodies. With some fluid meters, p_s and p_0 are measured, and V_0 is then calculated by Equation (33). Fluid meters will be discussed in detail in Chapter 8.

Applying the energy equation between points O and B in Fig. 39 gives

$$\frac{p_0}{w} + \frac{V_0^2}{2g} = \frac{p_B}{w} + \frac{V_B^2}{2g}. \tag{34}$$

If p_0 and V_0 are known, p_B can be calculated if V_B is known. V_B might be determined by a kinematic study, as indicated in Chapter 3.

Example 1. A submarine moves through salt water at a depth of 55 feet with a speed of 12 miles per hour. Determine the gage pressure on the nose of the submarine. Specific weight of the water is 64.0 pounds per cubic foot.

$$p_s = p_0 + \tfrac{1}{2}\rho V_0^2;$$

$$p_s = 55(64) + \frac{1}{2}\left([illegible] \frac{.0}{2}\right)\left[\frac{12 \times 5280}{3600}\right]^2 \text{ pounds per square foot;}$$

$$p_s = [illegible] = 26.6 \text{ pounds per square inch}$$

Example 2. In a centrifugal pump test the discharge gage reads 100 pounds per square inch, and the suction gage reads 5 pounds per square inch. Both gages indicate pressure above atmospheric. The gage centers are at the same level. The diameter of the suction pipe is 3 inches, and the diameter of the discharge pipe is 2 inches. Oil (specific gravity = 0.85) is being pumped at the rate of 100 gallons per minute. Assuming no friction losses, what is the power delivered to the pump?

Let the suction be section 1 and the discharge be section 2. 1 gallon = 231 cubic inches. The energy equation for this case is

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + \text{work} = \frac{p_2}{w} + \frac{V_2^2}{2g}.$$

The rate of discharge Q is 100(231)/60(1728) = 0.223 cubic feet per second. From the continuity equation $A_1V_1 = A_2V_2 = Q$,

$$V_1 = \frac{0.223}{\frac{\pi}{4}\left(\frac{3}{12}\right)^2} = 4.54 \text{ feet per second,}$$

$$V_2 = \frac{0.223}{\frac{\pi}{4}\left(\frac{2}{12}\right)^2} = 10.22 \text{ feet per second,}$$

$$\text{work} = \frac{144}{62.4(0.85)}(100 - 5) + \frac{1}{64.4}((10.22)^2 - (4.54)^2)$$

$$= 259.3 \text{ foot-pounds per pound.}$$

Since 1 horsepower = 550 foot-pounds per second,

$$\text{horsepower} = \frac{259.3(0.223)62.4(0.85)}{550} = 5.58.$$

PLE 3. Air enters a piece of equipment at a pressure of 14.7 pounds per square inch absolute, and a temperature of 60° Fahrenheit. Entrance area equals exit area = 1 square foot. At exit the pressure is 14.5 pounds per square inch absolute, and the temperature is 200° Fahrenheit. Entrance velocity is 30 feet per second. This equipment delivers 5 horsepower to another machine. Assume $z_1 = z_2$. $c_p = 0.24$. Find the heat added.

The energy equation for this case is

$$q + \text{work} = h_2 - h_1 + \frac{V_2^2 - V_1^2}{2g(778)},$$

and

$$h_2 - h_1 = c_p(T_2 - T_1) = 0.24(200 - 60).$$

V_2 can be determined by means of the continuity equation and the equation of state:

$$\frac{A_1V_1}{v_1} = \frac{A_2V_2}{v_2}, \qquad \frac{p_1v_1}{p_2v_2} = \frac{T_1}{T_2}.$$

Therefore

$$V_2 = \left(\frac{T_2p_1}{T_1p_2}\right) V_1 = \frac{(200 + 460)}{(60 + 460)} \left(\frac{14.7}{14.5}\right) 30 = 38.6 \text{ feet per second.}$$

Since $p_1v_1 = RT_1$,

$$v_1 = \frac{53.3(460 + 60)}{14.7(1} \quad 13.09 \text{ cubic feet per pound.}$$

$$\text{Weight flow} = \frac{A_1V_1}{v_1} = \frac{30}{13.09} = 2.29 \text{ pounds per second.}$$

Since 1 horsepower-hour = 2545 B.t.u.,

$$\text{work} = \frac{5(2545)}{2.29(3600)} \text{ B.t.u. per pound,}$$

$$q = \frac{5(2545)}{2.29(3600)} + 0.24(200 - 60) + \frac{(38.6)^2 - (30)^2}{2g(778)}$$
$$= 35.15 \text{ B.t.u. per pound added to fluid.}$$

SELECTED REFERENCES

Energy Transfer Between a Fluid and a Rotor for Pump and Turbine Machinery by S. A. Moss, C. W. Smith, and W. R. Foote. *A.S.M.E. Transactions*, August, 1942, vol. 64, no. 6, page 567.

A Thermodynamic Analysis of the Steady Flow of Fluids by C. H. Berry. *Mechanical Engineering*, November, 1929, page 816.

PROBLEMS

43. An airship flies through still standard air at 80 miles per hour. What is the gage pressure on the nose?

44. A Thomas flow meter is inserted in a section of horizontal pipe 6 inches in diameter. This meter consists of two thermometers and an electrical heating element. Carbon dioxide flows through the thermally insulated pipe. At section 1 the temperature is 70° Fahrenheit and the absolute pressure is 30.2 inches of mercury. At section 2 the temperature is 80° Fahrenheit and the absolute pressure is 29.92 inches of mercury. Between sections the electrical heating element adds a total of 100 B.t.u. per minute to the gas. $c_p = 0.21$. Consider the fluid incompressible if the density variation is less than 4 per cent. Find the weight of gas flowing per minute.

45. A fluid flows adiabatically through a horizontal diverging nozzle. The entrance velocity is 1000 feet per second. Vapor tables give 1200 B.t.u. per pound for exit enthalpy, and 1322.5 B.t.u. per pound for entrance enthalpy. What is the exit velocity?

46. Brine (specific gravity = 1.20) flows through a pump at 2000 gallons per minute. The pump inlet is 12 inches in diameter. At the inlet the vacuum is 6 inches of mercury. The pump outlet, 8 inches in diameter, is 4 feet above the inlet. The outlet pressure is 20 pounds per square inch gage. What power does the pump add to the fluid?

47. Air enters a horizontal pipe 14 inches in diameter at an absolute pressure of 14.8 pounds per square inch and a temperature of 60° Fahrenheit. At exit the

pressure is 14.5 pounds per square inch absolute. Entrance velocity is 35 feet per second, exit velocity is 50 feet per second. $c_p = 0.24$, $c_v = 0.17$. What is the heat added or abstracted?

48. A liquid is being heated in a vertical tube of uniform diameter, 50 feet long. The flow is upward. At entrance the average velocity is 3.5 feet per second, the absolute pressure is 50 pounds per square inch, and the specific volume is 0.017 cubic feet per pound. At exit the absolute pressure is 48 pounds per square inch, and the specific volume is 0.90 cubic feet per pound. The increase in internal energy is 10 B.t.u. per pound. Find the heat added.

49. A machine has fluid supplied to it at a rate of 250,000 pounds per hour. The fluid enters the machine with an enthalpy of 1290 B.t.u. per pound and a velocity of 5000 feet per minute at a point 15 feet above the exit, where it leaves with an enthalpy of 920 B.t.u. per pound and a velocity of 10,000 feet per minute. The machine delivers 36,000 horsepower. Find the heat added or abstracted in B.t.u. per hour.

50. Where would be a desirable place for locating ventilating openings along the length of a streetcar if: (a) draft upon starting and stopping is to be avoided, and (b) maximum exchange of air is desired?

51. An air compressor receives air at a pressure of 14.3 pounds per square inch absolute, with a specific volume of 2 cubic feet per pound. The fluid flows steadily through the machine and is discharged at 110 pounds per square inch absolute, with a specific volume of 0.40 cubic feet per pound. The initial internal energy is 10 B.t.u. per pound, whereas the final internal energy is 48 B.t.u. per pound. The cooling water around the cylinder carries away 34 B.t.u. per pound. Neglecting changes in kinetic energy and potential energy, what is the work? Is this work done on or by the fluid?

CHAPTER 5

Viscosity or Internal Friction

> The resistance arising from the want of lubricity in the parts of a fluid, is, other things being equal, proportional to the velocity with which the parts of the fluid are separated from one another.—SIR ISAAC NEWTON.[1]

The flow of any real fluid gives rise to tangential frictional forces which are called *viscous* forces. The action of such internal shearing forces results in a degradation of mechanical energy into heat or unavailable thermal energy.

39. Definition of viscosity

When a shear stress is applied to an elastic solid material, there is a definite angular deformation which is proportional to the stress. The

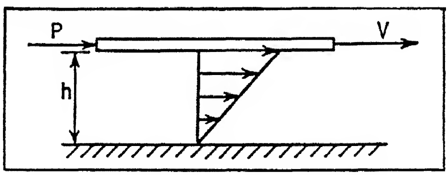

FIG. 40. Flow between parallel plates. The lower plate is stationary.

modulus of elasticity in shear, G, is defined as the ratio of the shear stress to the shear strain (angular deformation in radians). G is a characteristic property of the material; for steel G is about 11.5×10^6 pounds per square inch. The shear modulus G for a fluid is zero. *Viscosity* is a similar modulus, or factor of proportionality, which is a characteristic property of a fluid. Viscosity, however, differs from G in one important respect, in that viscosity involves a *rate* of shearing strain.

Fluid fills the space between the two parallel plates in Fig. 40. The upper plate moves with the velocity V, while the lower plate is stationary. A very thin layer of fluid adheres to the lower plate; the velocity of this layer is zero. A thin layer of fluid adheres to the upper plate; this layer of fluid has the velocity V. It will be assumed that the fluid flows in parallel layers or laminas, and that there are no secondary irregular fluid

[1] *Sir Isaac Newton's Mathematical Principles of Natural Philosophy and His System of the World* by Florian Cajori. University of California Press, Berkeley, California, 1934.

motions and fluctuations superimposed on the main flow. Such flow is called *laminar.* P is the force required to maintain the flow, to slide the fluid layers relative to each other by overcoming the internal fluid resistance. If A is the area of the plate in contact with the fluid, then the shear stress is $P/A = \tau$ (Greek letter tau).

The linear velocity distribution in the fluid is shown in Fig. 40. The *rate of shearing strain* of the fluid is V/h. During each unit of time there is an angular change equal to V/h. The coefficient of viscosity of the fluid is defined as follows:

$$\text{viscosity} = \frac{\text{shearing stress}}{\textit{rate} \text{ of shearing strain}} \tag{35}$$

Sometimes the foregoing term is called *absolute viscosity.* Probably a better term would be *dynamic viscosity.* Its symbol is μ (Greek letter mu):

$$\text{dynamic viscosity} = \mu = \frac{P/A}{V/h} = \frac{\tau}{V/h},$$

$$\tau = \mu \frac{V}{h}. \tag{36}$$

If the velocity distribution is nonlinear, as indicated in Fig. 41, the shear stress varies from point to point in the fluid. The shear stress at a point is

$$\tau = \mu \frac{du}{dy}, \tag{37}$$

where du is the infinitesimal velocity change in the infinitesimal distance dy. The ratio du/dy is the velocity gradient or rate of shearing strain. *Fluidity* is defined as the reciprocal of dynamic viscosity.

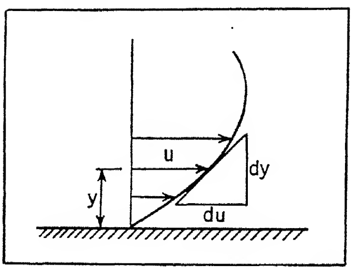

FIG. 41. Nonlinear velocity distribution.

40. Units of dynamic viscosity

The units of dynamic viscosity can always be determined by reference to the fundamental definition given in Equation (35). Let the symbols F, M, L, and T represent the primary or fundamental dimensions force, mass, length, and time respectively. Area is then represented by L^2, and velocity by LT^{-1}. Since force equals mass times acceleration, the symbol for force F can be replaced by its equivalent MLT^{-2}. The rate of shearing strain has the dimensions of T^{-1}, for the ratio of velocity to length has the dimensions of T^{-1}. Then the

$$\text{dimensions of dynamic viscosity are } \frac{FT}{L^2}$$

or the

$$\text{dimensions of dynamic viscosity are } \frac{MLT^{-2}T}{L^2} \text{ or } \frac{M}{LT}.$$

Dynamic viscosity may be expressed in pounds-second per square foot or slugs per foot-second:

$$1\,\frac{\text{pound-second}}{\text{square foot}} = 1\,\frac{\text{slug}}{\text{foot-second}}.$$

In the metric system, dynamic viscosity may be expressed in dynes-second per square centimeter or grams per centimeter-second.

$$1\,\frac{\text{dyne-second}}{\text{square centimeter}} = 1\,\frac{\text{gram}}{\text{centimeter second}} = 1 \text{ poise}.$$

The term *poise* is in honor of Poiseuille, a French scientist. The centipoise, or 0.01 poise, is a common unit. The dynamic viscosity of water at 20° centigrade is approximately 1 centipoise.

Some conversions between the American and metric systems are as follows:

$$1 \text{ inch} = 2.54 \text{ centimeters}; \qquad 1 \text{ dyne} = 2.248 \times 10^{-6} \text{ pounds}.$$

Then

$$\begin{aligned} 1 \text{ poise} &= 1\,\frac{\text{dyne-second}}{\text{centimeter squared}} \\ &= 2.248 \times 10^{-6}(2.54 \times 12)^2\,\frac{\text{pounds-second}}{\text{foot squared}} \\ 1 \text{ poise} &= 2.089 \times 10^{-3}\,\frac{\text{pounds-second}}{\text{foot squared}} \\ &= 2.089 \times 10^{-3}\,\frac{\text{slugs}}{\text{foot-second}} \end{aligned}$$

41. Kinematic viscosity

Kinematic viscosity is defined as the ratio of dynamic viscosity to density.

$$\text{kinematic viscosity} = \nu(\text{nu}) = \frac{\mu}{\rho}.$$

Dimensions of ν are

$$\frac{ML^3}{LTM} = \frac{L^2}{T}.$$

Kinematic viscosity can be expressed in terms of feet squared per second, or centimeters squared per second. In honor of Sir George Stokes, an English scientist, 1 centimeter squared per second is called a *stoke*.

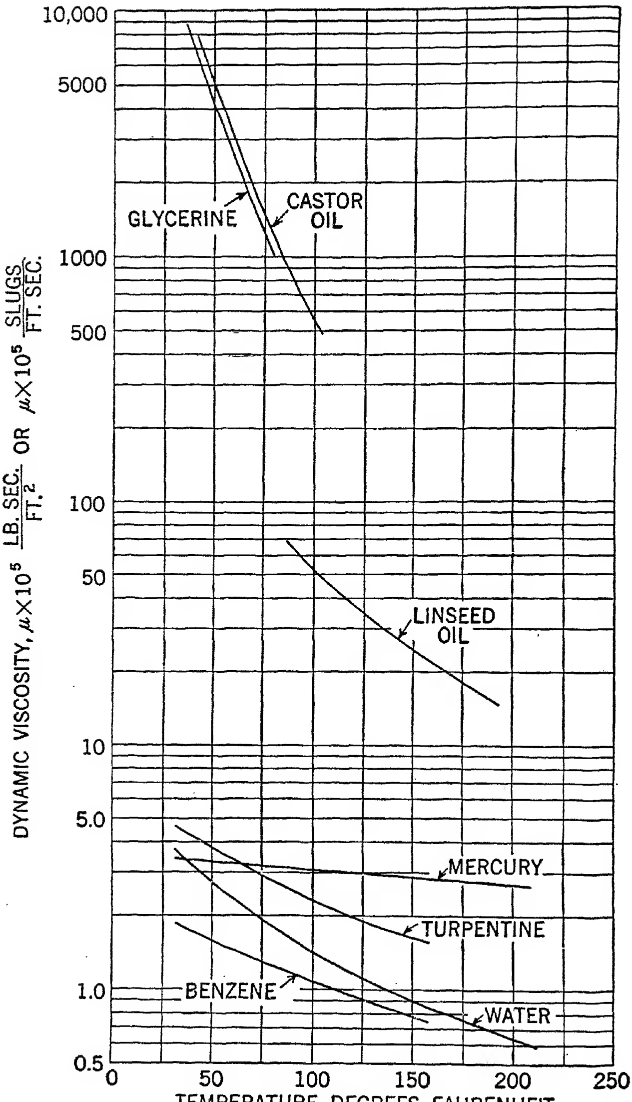

Fig. 42. Dynamic viscosity of some liquids.[2]

[2] Data for water from *Properties of Ordinary Water-Substance* by N. E. Dorsey, Reinhold Publishing Co., New York, 1940. Data for other liquids from *Smithsonian Physical Tables*, Eighth Revised Edition, Smithsonian Institution, Washington, D. C., 1933.

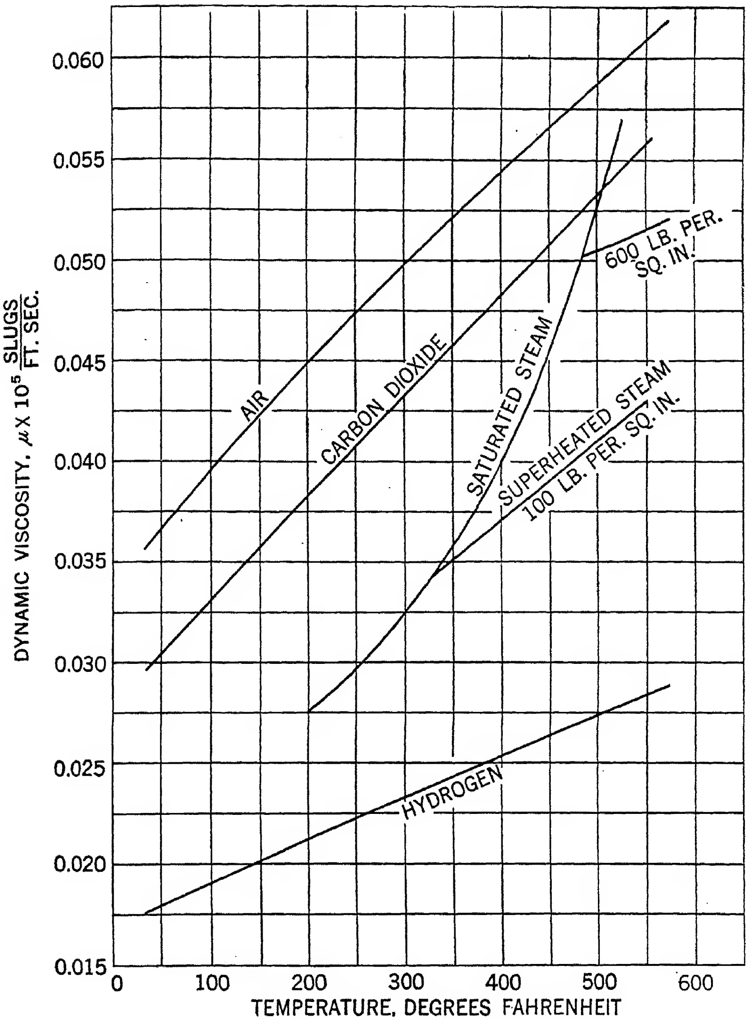

FIG. 43. Dynamic viscosity of some gases, at atmospheric pressure, and of steam.[3]

[3] Data for steam from *The Viscosity of Superheated Steam*, by G. A. Hawkins, H. L. Solberg, and A. A. Potter, *A.S.M.E. Transactions*, vol. 26, no. 8, November, 1940, page 677. Other data adapted from *Der Chemie-Ingenieur* by M. Jakob and S. Erk, Akademische Verlagsgesellschaft M.B.H., Leipzig, 1933, vol. 1, part 1, page 73.

42. Numerical values of viscosity

Figure 42 shows a general feature for liquids, that the dynamic viscosity decreases as the temperature increases. Figure 43 shows a general feature for gases, that the dynamic viscosity increases as the temperature increases. The exact nature of the internal friction between adjacent layers of moving fluids has not been established at the present time. The explanation of the viscosity of gases is probably more fully developed than that for liquids.

From the point of view of the kinetic theory of gases, viscous action may be regarded as due to a process of momentum exchange or momentum diffusion between adjacent gas layers which have different velocities. This momentum exchange is caused by molecular motion, and results in a tendency to reduce the relative motion between neighboring layers. This theory agrees with the fact that the dynamic viscosities of gases increase with temperature increase; as the temperature rises the molecular activity increases, and the resistance to relative motion increases.

The effect of small pressure changes on the dynamic viscosity of fluids is usually considered negligible for ordinary conditions. Table 3 gives some values at 59° Fahrenheit and 29.92 inches of mercury. The viscosities of some common lubricants at various temperatures are given in the chapter on lubrication, Chapter 15.

TABLE 3
SOME VALUES OF VISCOSITY AT STANDARD CONDITIONS

Fluid	Dynamic viscosity, $\frac{\text{slugs}}{\text{ft.-sec.}}$	Kinematic viscosity, $\frac{\text{ft.}^2}{\text{sec.}}$
Air	0.0373×10^{-5}	1.57×10^{-4}
Water	2.391×10^{-5}	1.233×10^{-5}
Castor oil	3160×10^{-5}	1.692×10^{-2}

43. Fluids and other substances

The behavior of materials under shear stress seems to offer the best means for distinguishing between a fluid and a solid. A substance is a fluid if it is continuously and permanently deformed by a shear stress, no matter how small the stress. There are substances—for example tar, sealing wax, and some glues—which appear to behave like solids under certain circumstances; these substances, however, may be classified as very viscous liquids with very low rates of deformation. At times some plastic substances are confused with fluids. A soft or plastic substance like lead, soap, sewage sludge, clay slurry, or a tallow candle may flow, but it only flows after a certain minimum stress has been exceeded. Such a plastic substance is, therefore, not a true fluid as normally defined.

Figure 44, a plot of shearing stress against rate of shearing strain, brings out these distinctions. Figure 44 shows a straight line passing through the origin for fluids. The ratio between shearing stress and rate of shearing strain is the same for all rates of shearing strain. The dynamic viscosity of a particular fluid is uniquely determined by the temperature and pressure, regardless of the rate of shearing strain. For the plastic, as shown in Fig. 44, the minimum stress necessary before flow starts is sometimes called the *yield* value.

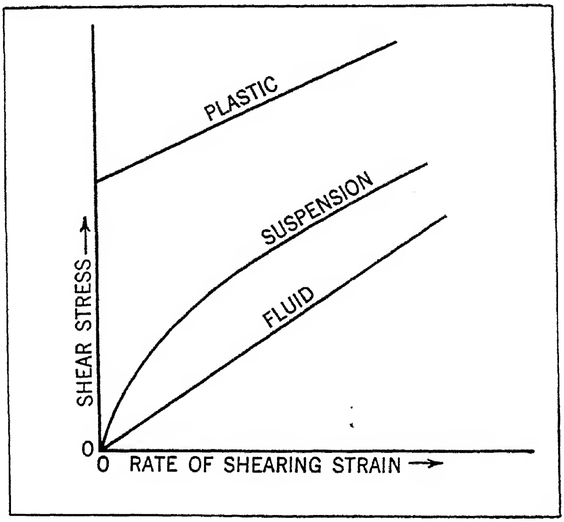

FIG. 44. Types of diagrams of shear stress against rate of shearing strain for various classes of substances.

In many industrial processes the engineer is concerned with the movement of a combination consisting of a fluid and suspended particles. Sometimes the prediction of friction losses for such suspensions becomes an important and difficult problem. There is a scarcity of experimental data on the flow properties of suspensions. There is some question as to the meaning and measurement of the "viscosity" of a suspension. Taking the meaning as defined by Equation (35), experiments show that a suspension may have a "viscosity" considerably higher than that of the fluid carrier. Further, the "viscosity" of a suspension may be different for different rates of shearing strain. This introduces a complication not found with fluids. This book will be concerned only with fluids, namely with substances which have the same ratio between shear stress and rate of shearing strain for all rates of shearing strain.

SELECTED REFERENCES

Fluidity and Plasticity by E. C. Bingham, McGraw-Hill Book Co., New York, 1922.

Some Physical Properties of Water and Other Fluids by R. L. Daugherty, *A.S.M.E. Transactions*, vol. 57, 1935, page 193.

The Viscosity of Liquids by E. Hatschek, G. Bell and Sons, Ltd., London, 1928.

PROBLEMS

52. Castor oil at 59° Fahrenheit fills the space between two parallel horizontal plates which are $\frac{3}{8}$ inch apart. If the upper plate moves with a velocity of 5 feet per second, and the lower one is stationary, what is the shear stress in the oil?

53. A liquid has a dynamic viscosity of 1.85 centipoises and a specific gravity of 1.046. What are the dynamic and kinematic viscosities in American units, and the kinematic viscosity in stokes?

54. The kinematic viscosity of an oil is 0.020 foot squared per second, and the specific gravity is 0.87. Determine the dynamic viscosity.

55. A long vertical cylinder 3.00 inches in diameter rotates concentrically inside a fixed tube having a diameter of 3.02 inches. The uniform annular space between tube and cylinder is filled with water at 59° Fahrenheit. Assuming a linear velocity distribution, what is the resistance to motion at a relative velocity of 3 feet per second, for a 6-inch length of cylinder? What would be the resistance if the fluid were castor oil at 59° Fahrenheit?

CHAPTER 6

Dimensional Analysis and Dynamic Similarity

> The discovery and use of scientific reasoning by Galileo was one of the most important achievements in the history of human thought.—ALBERT EINSTEIN and LEOPOLD INFELD.[1]

Modern fluid mechanics is based upon a healthy combination of physical analysis and experimental observations. The general objective is to provide dependable practical results, and a thorough understanding of fundamental flow features. Many flow phenomena are so complex that a purely mathematical solution is impractical, incomplete, or impossible. The present chapter discusses dimensional analysis and dynamic similarity, two tools which have proved very useful in the organization, correlation, and interpretation of experimental data. The following chapters will take advantage of these tools.

DIMENSIONAL ANALYSIS

44. Dimensions

Dimensional analysis is a mathematical method useful in: (a) checking equations; (b) changing units; (c) determining the general form of a physical equation; and (d) planning systematic experiments. The following discussion deals mainly with the problem of determining the general form of a physical equation.

Dimensional analysis results in a sound, orderly arrangement of the variable physical quantities involved in a problem. Reference to experimental data must be made in order to obtain the necessary constants or coefficients for a complete numerical expression. Although the following examples deal with fluid flow, it is to be emphasized that the same method of approach can be used for problems in rigid-body mechanics, heat transfer, thermodynamics, electricity, and other fields of pure and applied physics.

The first step in treating a problem is to list all the variables involved. This listing may be the result of experience or judgment. The form of the equation which results from a dimensional analysis is no more accurate or complete than the original listing of variables. The next step, the dimensional analysis, can be made by following a formal procedure.

It should be noted that the same word, *dimension*, is used to signify both the numerical magnitude of a measurement and its dimensional class

[1] *The Evolution of Physics.* Simon and Schuster, New York, 1938, page 7.

or category. The subject of mechanics deals quantitatively with physical phenomena involving mass, force, velocity, energy, density, and many other measurable characteristics. An arbitrary dimensional unit might be ascribed to each measurable characteristic; units of area, volume, and energy might be devised. For example, speed might be expressed in arbitrary speed units, as by the Beaufort wind scale employed by sailors and meteorologists. A Beaufort number 3 designates a gentle breeze, and a Beaufort number 9 designates a strong gale. Such a creation of dimensional units for all measurable characteristics, however, is not generally convenient or necessary.

Let the letter L be the dimensional symbol for length, M the dimensional symbol for mass, F for force, and T for time. In the American system, which is followed in this book, the foot is a unit of length, the slug is a unit of mass, the pound is a unit of force, and the second is a unit of time. In the metric system, the centimeter is a unit of length, the gram is a unit of mass, the dyne is a unit of force, and the second is a unit of time.

If length is arbitrarily selected as the fundamental or primary dimension of space, units of area and volume can be devised as secondary or derived units. Area is represented by the dimensional symbol L^2; volume is represented by the dimensional symbol L^3. A measurement of the volume of a body expressed in terms of cubic inches or cubic yards does not make any difference in the actual volume contained in the body. Units are arbitrary. The volume dimension depends upon the third power of the primary dimension length, regardless of the dimensional units employed. If time is also taken as a primary dimension, then the unit of linear velocity can be arbitrarily taken as a secondary or derived unit. Linear velocity is defined in terms of length and time. The dimensional symbol for linear velocity is LT^{-1}. Similarly, the dimensional symbol for linear acceleration becomes LT^{-2}.

The selection of force or mass as a primary dimension is solely a matter of convenience. Either one could be selected; neither is "absolute." If one is selected, the other becomes a secondary or derived dimension. The dimensional relation between these two quantities can be found from the relation:

force equals mass times acceleration.

In dimensional symbols, $F = MLT^{-2}$ or $M = FT^2L^{-1}$.

For ordinary problems in mechanics three primary dimensions are necessary and sufficient. L, T, and M might be used as one combination of primary dimensions. L, T, and F might be used as an alternate combination.[2] Table 4 gives the dimensions of some physical quantities using the M, L, and T system.

[2] In heat transfer the fundamental dimensions are usually taken as length, time,

TABLE 4
DIMENSIONS OF SOME PHYSICAL QUANTITIES

Quantity	*Dimensions*
Length	L
Mass	M
Time	T
Force (F)	MLT^{-2}
Area	L^2
Volume	L^3
Linear velocity	LT^{-1}
Linear acceleration	LT^{-2}
Angular measure	None
Angular velocity	T^{-1}
Torque or Moment (FL)	ML^2T^{-2}
Density	ML^{-3}
Specific weight (FL^{-3})	$ML^{-2}T^{-2}$
Dynamic viscosity	$ML^{-1}T^{-1}$
Kinematic viscosity	L^2T^{-1}
Pressure or Stress (FL^{-2})	$ML^{-1}T^{-2}$
Work, Energy, or Heat	ML^2T^{-2}

45. Buckingham's π theorem

The complete definition of a physical quantity implies both an algebraic equation and a corresponding dimensional equation. For example, the algebraic equation for linear velocity can be written as $V = dl/dt$. The corresponding dimensional equation states that linear velocity has the dimensional symbol LT^{-1}. An equation expressing a functional relation between physical quantities is an algebraic equation; it does not show explicitly the units in which each of the various quantities are measured. A corresponding dimensional equation can be written, to show the manner in which the dimensions of these units are combined in each term of the algebraic equation.

This book will follow the customary practice in engineering work of dealing solely with physical equations which are dimensionally homogeneous. All the terms of a dimensionally homogeneous equation have the same dimensions. Dissimilar quantities cannot be added or subtracted to form a true physical relation. A nonhomogeneous equation is:

$$\text{land} + \text{cows} = \text{farm},$$

which may have some meaning, but is not the type of relation under consideration.

It is sound and most convenient to have a physical equation arranged with the variables in dimensionless groups or combinations. Variables can be organized in the smallest number of significant groups by means of Buckingham's π (Greek letter pi) theorem.

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temperature, mass, and heat. Heat may be used as a fundamental dimension as well as mass, as long as no interchange of mechanical energy and heat takes place. If appreciable conversion of mechanical energy into heat takes place, then under no condition may heat be regarded as a fundamental dimension.

Let $A_1, A_2, A_3, \cdots A_n$ be n physical quantities which are involved in some physical phenomenon. Examples of these physical quantities are velocity, viscosity, and density. Let m be the number of all the primary or fundamental units (such as length, mass, and time) involved in this group of physical quantities. The physical equation, or the functional relation between these quantities, can be written as

$$f(A_1, A_2, A_3, \cdots A_n) = 0.$$

The π theorem states that the foregoing relation can be written as

$$\phi(\pi_1, \pi_2, \cdots \pi_{n-m}) = 0$$

where each π is an independent dimensionless product of some of the A's. Note that the number of terms in the physical equation has been reduced from n to $n - m$. A formal proof of the π theorem is somewhat abstract, and will not be given here.[3] The purpose of the present discussion is to provide information useful in gaining a working knowledge for engineering applications.

If there are three primary dimensions, as for ordinary problems in mechanics, $m = 3$. In this case, the maximum number of independent products can be obtained by expressing the π products in the form

$$\begin{aligned} \pi_1 &= A_1^{x_1}A_2^{y_1}A_3^{z_1}A_4 \\ \pi_2 &= A_1^{x_2}A_2^{y_2}A_3^{z_2}A_5 \\ \cdots & \quad \cdots \\ \pi_{n-m} &= A_1^{x_{n-m}}A_2^{y_{n-m}}A_3^{z_{n-m}}A_n \end{aligned}$$

There will be $m + 1$ variables in each term; only one variable need be changed from term to term. In each product there are three unknown exponents in x, y, and z. A consideration of the three primary dimensions (for example, L, T, and F or M) yields three separate equations; a simultaneous solution of these three equations gives numerical values for the three exponents. Similarly, if $m = 4$, each π product can be expressed in terms of five A's, with four of the A's raised to unknown powers.

The following gives a summary in words, and some suggestions for a formal procedure.

If there are n physical variables in a particular problem, and m fundamental units (such as length, mass, and time), then the physical equation can be expressed in a form involving $(n - m)$ dimensionless ratios. These dimensionless ratios can be found by using the following logical and efficient procedure:

> (a) Select from among the list of variables a number of variables equal to the number of fundamental units and including all of the fundamental units.

[3] A formal proof is given in *On Physically Similar Systems; Illustrations of the Use of Dimensional Equations* by E. Buckingham, *Physical Review*, vol. 4, 1914, page 345.

(b) Set up dimensional equations combining the variables selected in (a) with each of the others in turn.

The general method of approach can best be explained by several examples.

46. Illustrations of dimensional analysis

EXAMPLE 1. Consider a rotating shaft in a well-lubricated bearing. The problem is to determine the form of the equation giving the frictional resistance in terms of pertinent variables. The case is taken in which the following variables are involved:

Variable	Symbol	Dimensions of variable
Tangential friction force	R	MLT^{-2}
Force normal to shaft	P	MLT^{-2}
Shaft revolutions per unit time	N	T^{-1}
Viscosity of lubricant	μ	$ML^{-1}T^{-}$
Shaft diameter	D	L

Since there are five variables ($n = 5$) and three fundamental units ($m = 3$), the physical equation has two dimensionless ratios. Let π_1 and π_2 represent these ratios. Dimensional equations are written combining the three variables P, N, and D [selected as suggested in (a) of Article 45] with each of the remaining variables in turn. Thus

$$\pi_1 = P^{x_1}N^{y_1}D^{z_1}R, \qquad \pi_2 = P^{x_2}N^{y_2}D^{z_2}\mu.$$

The six exponents are found from dimensional considerations. Substituting the dimensions for the symbols in the equation of π_1 gives

$$\left(\frac{ML}{T^2}\right)^{x_1}\left(\frac{1}{T}\right)^{y_1}L^{z_1}\frac{ML}{T^2} = L^0M^0T^0,$$

where $L^0M^0T^0$ represents the fact that the π_1 ratio is dimensionless or has zero dimensions. Solving for each dimension separately gives

$$\begin{array}{lll} M & x_1 + 1 = 0 & x_1 = -1 \\ L & x_1 + z_1 + 1 = 0 & z_1 = 0 \\ T & -2x_1 - y_1 - 2 = 0 & y_1 = 0 \end{array}$$

Therefore,

$$\pi_1 = \frac{R}{P}.$$

which is easily checked. R/P is commonly called a friction coefficient f.

Substituting the dimensions for the symbols in the equation for π_2 gives

$$\left(\frac{ML}{T^2}\right)^{x_2}\left(\frac{1}{T}\right)^{y_2}L^{z_2}\frac{M}{LT} = L^0M^0T^0.$$

$$\begin{array}{lll} M & x_2 + 1 = 0 & x_2 = -1 \\ L & x_2 + z_2 - 1 = 0 & z_2 = 2 \\ T & -2x_2 - y_2 - 1 = 0 & y_2 = 1 \end{array}$$

Thus

$$\pi_2 = \frac{N\mu D^2}{P}.$$

The physical equation has the form

$$\pi_1 = \phi(\pi_2) \qquad \text{or} \qquad f = \phi\left(\frac{N\mu D^2}{P}\right).$$

where ϕ means "some function of"; that is, the friction coefficient f is some function of the dimensionless ratio $N\mu D^2/P$. The foregoing form is used in lubrication studies.[4] Experimental data are necessary in order to determine the exact nature of the functional relation; such data are available in current lubrication literature. Dimensional analysis provides a systematic and efficient approach. Practical engineering work has shown that the result is of aid in the design and prediction of performance of lubricated bearings.

R, N, and D were selected to give *one* solution. Another set of three variables could have been selected to give *another* solution. For example, the dimensionless ratio P/R, from a dimensional point of view, it just as satisfactory and sound as its reciprocal. Custom, however, has arbitrarily established the ratio R/P as a friction coefficient.

EXAMPLE 2. The problem is to develop an expression for the thrust of a screw propeller completely immersed in a fluid. It is judged that the following variables are involved:

Variable	Symbol	Dimensions of variable
Thrust (axial force).........	P	MLT^{-2}
Propeller diameter..........	D	L
Velocity of advance.........	V	LT^{-1}
Revolutions per unit time...	N	T^{-1}
Gravitational acceleration....	g	LT^{-2}
Density of fluid............	ρ	ML^{-3}
Kinematic viscosity of fluid..	ν	L^2T^{-1}

Since there are seven variables and three fundamental units, there are four dimensionless ratios. Dimensional relations are written combining D, V, and ρ with each of the remaining variables in turn:

$$\begin{aligned}
\pi_1 &= D^{x_1}V^{y_1}\rho^{z_1}P,\\
\pi_2 &= D^{x_2}V^{y_2}\rho^{z_2}N,\\
\pi_3 &= D^{x_3}V^{y_3}\rho^{z_3}g,\\
\pi_4 &= D^{x_4}V^{y_4}\rho^{z_4}\nu.
\end{aligned}$$

The twelve exponents in x, y, and z are determined such that each π function is dimensionless. The result is

$$\pi_1 = \frac{P}{D^2V^2\rho}, \qquad \pi_2 = \frac{DN}{V}, \qquad \pi_3 = \frac{Dg}{V^2}, \qquad \pi_4 = \frac{\nu}{DV}.$$

[4] For further discussion see the chapter on lubrication, Chapter 15.

This dimensional analysis gives *one* solution to the problem. There are *other* solutions each involving different dimensionless groupings of the variables. The above indicates that the form of the physical equation involves the four ratios $\pi_1, \pi_2, \pi_3,$ and π_4. One solution of a variety can be written as

$$\pi_1 = \phi(\pi_2, \pi_3, \pi_4),$$

meaning that π_1 is some function of the other three π ratios, or

$$\frac{P}{D^2V^2\rho} = \phi\left(\frac{DN}{V}, \frac{Dg}{V^2}, \frac{\nu}{DV}\right).$$

Proper experiments would give the functional relation. The foregoing relation could be written as

$$P = KD^2V^2\rho\left(\frac{DN}{V}\right)^a\left(\frac{Dg}{V^2}\right)^b\left(\frac{\nu}{DV}\right)^c,$$

where the coefficient K and the exponents a, b, and c are to be found from experimental data.

47. Common dimensionless ratios

The last example includes one feature, in connection with the ratio $\pi_4 = \nu/DV$, which merits special attention. The reciprocal of this number is still a dimensionless ratio. The ratio DV/ν occurs frequently in fluid flow problems, and is significant in establishing criteria of flow. Its significance will be discussed more fully throughout this book. This particular number is called Reynolds' number, in honor of Osborne Reynolds. Other dimensionless ratios occur frequently in fluid mechanics and heat transfer, and have been given special names.

DYNAMIC SIMILARITY

48. Mechanically similar flows

In order to obtain information regarding the flow phenomena in or around a structure or machine, called the original or *prototype*, it is often convenient, economical, and sound engineering to experiment with a copy or *model* of the prototype. The model may be geometrically smaller than, equal to, or larger than the prototype in size. Model experiments for pumps, turbines, airplanes, ships, pipes, canals, and other structures and machines have resulted in savings which have more than justified the investments. Model tests provide an advantage in research, design, and performance-prediction work which cannot be obtained from theoretical calculations alone.

Certain laws of similarity must be observed in order to insure that the model-test data can be applied to the prototype. These laws, in turn, provide means for correlating and interpreting test data. For the flow around two bodies, as illustrated in Fig. 45, there is this question:

under what conditions is the flow around one body mechanically similar to the flow around the other? Mechanical similarity implies not only geometric similarity, but also similarity with respect to the forces acting, or dynamic similarity. The flow around two bodies can be similar only if the body shapes are geometrically similar; this geometric similarity is a necessary condition, but it is not a sufficient condition.

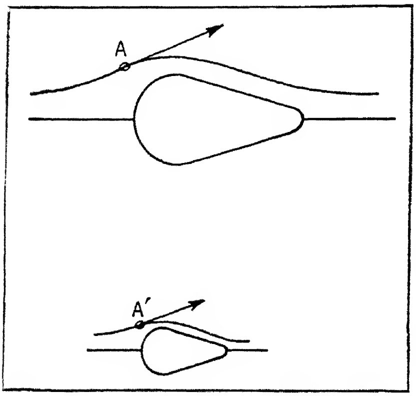

FIG. 45. Similar motions for a prototype and its model. A and A′ are corresponding points.

The streamline pattern for one body must also be similar to the streamline pattern for the other body. For corresponding points with respect to the bodies, the velocity direction in one flow must be the same as the velocity direction in the other flow. The velocity direction at any point in the field of flow is determined completely by the *ratio of the forces* acting on a fluid particle at the point. Therefore, mechanical similarity is realized when the ratio of the forces acting on a fluid particle in one flow is the same as the ratio of the forces acting at a corresponding point in the other flow.

Various laws of similitude could be devised, depending upon the type of forces acting. The present discussion will consider cases in which the following forces predominate:[5] (1) inertia forces, (2) viscous forces, (3) gravity forces, (4) pressure forces.

As pointed out in Chapter 2, the *magnitude* of the inertia force equals the product of the particle mass and particle acceleration. The direction of this inertia force is opposite to the direction of the acceleration of the particle.

It is customary to consider separately two cases, or combinations, each with only three forces, as:

(a) viscous, inertia, and pressure forces;
(b) inertia, gravity, and pressure forces.

In each case the pressure force is uniquely determined by the other two forces. In each case, specifying two of the forces automatically specifies the third force because the three forces are in equilibrium (recalling D'Alembert's principle). Therefore, in case (a) the significant pair of

[5] Elastic forces resulting from the compressibility of the fluid are discussed in Chap. 12.

forces can be taken as viscous and inertia, whereas in case (b) the significant pair of forces can be taken as inertia and gravity. Each combination will be treated in the next two articles.

49. Reynolds' number

For completely enclosed flow (pipes, many flow meters, fans, pumps, turbines), or for flow in which bodies are fully immersed in a fluid (vehicles, submarines, aircraft, structures) in such fashion that free surfaces do not enter into the consideration and gravity forces are balanced by buoyant forces, the inertia and viscous forces are the only ones which need to be taken into account. Mechanical similarity exists if, at points similarly located with respect to the bodies, the ratios of the inertia forces to the viscous forces are the same.

Since the product of mass multiplied by acceleration is proportional to

$$\text{volume} \times \text{density} \times \frac{\text{velocity}}{\text{time}},$$

the inertia force is proportional to

$$\frac{l^3\rho V}{t} = \frac{l^3\rho V}{l/V} = l^2\rho V^2.$$

where V is some characteristic velocity (for example the velocity of a body in a fluid or the average velocity over a fixed cross section of pipe), and l is some characteristic length (such as the diameter or length of a body, or the internal diameter of a pipe). The viscous force is proportional to τl^2, where τ is the viscous shear stress. Since $\tau = \mu \dfrac{du}{dy}$, the internal friction or viscous force is proportional to $\mu V l$. Then the dimensionless ratio $\dfrac{\text{inertia force}}{\text{viscous force}}$ is proportional to

$$\frac{l^2\rho V^2}{\mu V l} = \frac{\rho V l}{\mu} = \text{Reynolds' number.}$$

If viscous and inertia forces determine the flow for a prototype, then mechanical similarity between model and prototype is realized when the dimensionless Reynolds' number for the model equals the Reynolds' number for the prototype. In subsequent chapters, experimental data for various flow phenomena will be expressed as a function of Reynolds' number.

The nature of any particular flow of a real fluid may be judged to some extent from the corresponding Reynolds' number. A small Reynolds' number indicates that viscous forces predominate, whereas a large value of Reynolds' number indicates that inertia forces predominate. The laws of motion of fluids and the laws of resistance to motion are very different for these two cases.

EXAMPLE. Standard air flows through a pipe 32 inches in diameter with an average velocity of 6 feet per second. A model of this pipe 3 inches in diameter is constructed for water flow. What average velocity of water is necessary for the model flow to be dynamically similar to that of the prototype? Taking the diameter D of the pipe as a characteristic length,

$$\frac{V_1D_1}{\nu_1}\text{ (model)} = \frac{V_2D_2}{\nu_2}\text{ (prototype).}$$

Using values given in Table 3, Chapter 5, the velocity V_1 becomes

$$V_1 = V_2\left(\frac{D_2}{D_1}\right)\frac{\nu_1}{\nu_2} = 6\left(\frac{32}{3}\right)\left(\frac{1.233 \times 10^{-5}}{1.57 \times 10^{-4}}\right)$$

$$V_1 = 5.03 \text{ feet per second}$$

for the average velocity in the model pipe.

50. Froude's number

Gravity forces are taken into account in cases in which a fluid free surface plays an essential rôle, for example the surface waves produced by a ship or a seaplane hull, or the flow in an open channel. In the surface waves produced by a ship, the form of the liquid is displaced above and below the mean surface level, and hence a weight or gravity force is involved. A model of a ship will produce the same shape of surface waves as the full-size ship if the ratio of inertia force to gravity force is the same at similarly located points with respect to the model and the prototype. The gravity force for a fluid particle equals the particle mass times the gravitational acceleration g. The gravity force is proportional to $\rho l^3 g$, where l is some characteristic length, like the length of the ship. Then the dimensionless ratio $\dfrac{\text{inertia force}}{\text{gravity force}}$ is proportional to

$$\frac{l^2\rho V^2}{\rho l^3 g} = \frac{V^2}{lg}.$$

The shapes of the water waves produced by the ship model and the prototype will be similar if the value of V^2/lg is the same for both. This dimensionless ratio, or usually its square root, is called Froude's number:

$$\text{Froude's number} = \frac{V}{\sqrt{lg}}.$$

EXAMPLE. An ocean vessel 600 feet long is to travel 15 miles per hour. What would be the corresponding speed for a geometrically similar model 6 feet long, towed in water for studies of wave resistance?

$$\frac{V_1}{\sqrt{l_1g}}\text{ (model)} = \frac{V_2}{\sqrt{l_2g}}\text{ (prototype).}$$

$$= 15\sqrt{\tfrac{6}{600}} = 1.5 \text{ miles per hour.}$$

The application of a special similarity law depends upon the forces determining the particular flow. Frequently the flow depends not on one ratio of forces only, but on two or possibly three ratios. An example is the problem of the total resistance of a ship moving on a surface of water. The total resistance consists mainly of frictional resistance due to the viscous forces and of surface wave resistance due to gravity force. The total resistance of the ship thus depends on both Reynolds' number and Froude's number. Consider a ship model, smaller in size than the prototype, tested in water. Reynolds' law indicates a model speed higher than that of the prototype, whereas Froude's law indicates a model speed lower than that of the prototype. Theoretically, it may be possible to combine both laws of similarity by using two different liquids. Actually, however, no two practical liquids are available which have sufficiently different kinematic viscosities. In practice, ship-model tests are based on Froude's number, and viscous forces are taken into account indirectly by computation. The procedure will be outlined in the article on ship resistance in Chapter 10.

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PROBLEMS

56. Determine the form of an expression for the velocity of sound c in a gas. Assume that c depends upon the density ρ, pressure p, and dynamic viscosity μ.

57. Observations indicate that the resistance R which the air offers to an airplane wing depends mainly on some characteristic length l, the speed V of the wing, and the density and viscosity of the air. Find the form of the expression for the resistance.

58. Consider that the resistance R of a flat plate immersed in a fluid is dependent on the fluid density and viscosity, the velocity, and the width b and the height h of the plate. Develop an expression for the resistance of the plate in terms of these quantities.

59. It is judged that the performance of a lubricating oil ring depends upon the following variables: Q, quantity of oil delivered per unit time; D, inside diameter of the ring; N, shaft speed in revolutions per unit time; μ, oil viscosity;

ρ, oil density; w, specific weight of the oil; and S, the surface tension in air (force per unit length). Develop an expression for Q in terms of the other variables.

60. For a certain type of installation, observations show that the gases in the wake of a smokestack flow downward a certain distance, called the *downwash.* Assume that the pertinent variables involved in this phenomenon are: depth of downwash l, diameter of stack D, wind velocity V_1, gas velocity V_2, and gas temperature θ. Find a form of the expression for the downwash.

61. An airship model is tested in standard air at a speed of 100 feet per second. Calculate the towing speed of the same model when tested submerged in water under similar dynamic conditions.

62. An airplane model has linear dimensions that are one-twentieth those of the full-sized plane. It is desired to test the model in a wind tunnel with an air speed equal to the flying speed of full-size plane. What must be the pressure in the wind tunnel? Assume the same air temperature in each case.

63. A centrifugal pump is to handle castor oil at 59° Fahrenheit, at a speed of 1000 revolutions per minute. A model pump is exactly two times as large. Experiments are to be made with the model pump using standard air. Determine the speed of the air pump.

64. A submerged submarine is to move at 6 miles per hour. A model is one tenth as long as the prototype. What would be the model speed if the water were the same in both cases?

65. A propeller-type mixer, 9 feet in diameter, is to be used submerged in soybean oil at 30° centigrade (specific weight 56.5 pounds per cubic foot, viscosity 0.085 slugs per foot-second). Tests on a model, 6 inches in diameter, in water at 59° Fahrenheit show that the best operating speed for the model is 18 revolutions per minute. Estimate the best angular speed for the prototype.

66. An ocean vessel 500 feet long is to travel at 10 miles per hour. What would be the speed for a model 20 feet long, towed in water for studies of surface wave resistance?

CHAPTER 7

Flow of Incompressible Fluids in Pipes

A wide variety of applications in engineering practice involves the flow of some fluid through a pipe. Many pipe lines are seldom seen; nevertheless they play an important rôle because they transport fluids conveniently and economically. There are almost as many miles of pipe lines in this country as there are miles of railroads. The present chapter discusses the flow of incompressible or constant-density fluids in pipes, whereas Chapter 13 discusses the flow of compressible fluids in pipes. Both of these chapters deal with the flow of a fluid in a *closed* conduit, that is, one in which the fluid completely fills the pipe. A treatment of the flow in *open* channels, in which the liquid has a free surface, is presented in Chapter 14.

51. Laminar and turbulent flow

There are two different types of flow: laminar and turbulent. In laminar flow the fluid moves in layers, or laminas. In turbulent flow

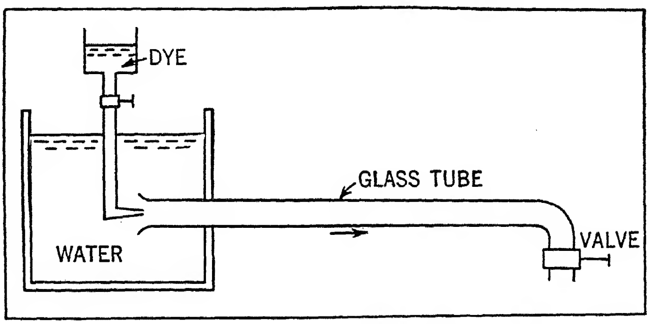

Fig. 46. Apparatus for investigating pipe flow.

there are secondary irregular motions and velocity fluctuations superimposed on the principal or average flow. As pointed out by H. L. Dryden, a common occurrence showing these types of flow is found in the rising column of smoke from a cigarette lying on an ashtray in a quiet room. For some distance the smoke rises in smooth filaments which may wave around but do not lose their identity; this flow is laminar. The filaments suddenly break up into a confused eddying motion at some

distance above the cigarette; this flow is turbulent. The transition between laminar and turbulent flows moves closer to the cigarette when the air in the room is disturbed. Turbulent flow is probably the more common type found in engineering applications.

The type of apparatus shown diagrammatically in Fig. 46 can be used to illustrate Reynolds' classical experiments.[1] Colored liquid entering

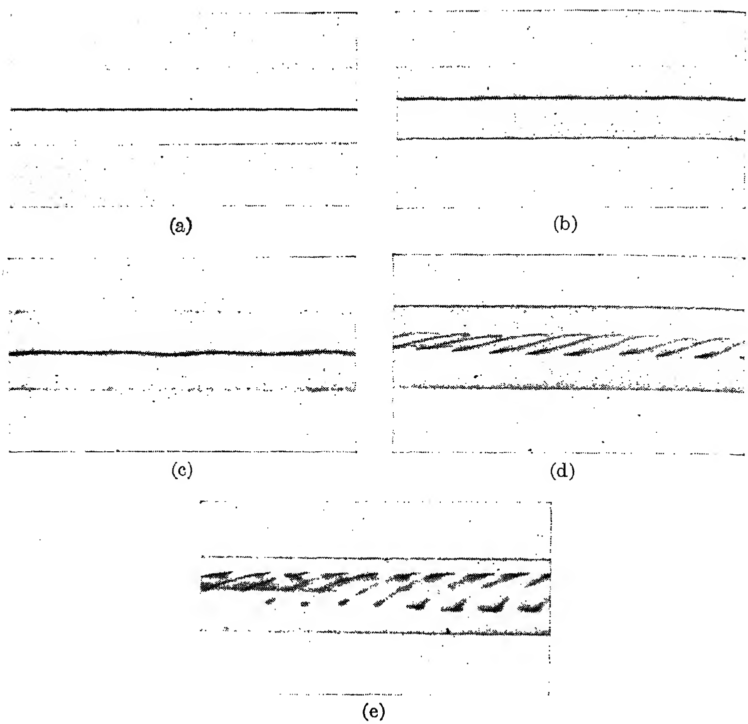

FIG. 47. Transition from laminar to turbulent flow for water in a glass tube, with increasing values of Reynolds' number.

the mouth of the glass tube moves along with the water, to indicate the nature of the flow. At low velocities the colored filament is straight and stable. At velocities above a critical value, the dye shows irregular patterns. In Fig. 47 are photographs taken with such an apparatus in the Purdue fluid-mechanics laboratory. The pictures are in the order

[1] *An Experimental Investigation of the Circumstances which Determine Whether the Motion of Water Will Be Direct or Sinuous, and of the Law of Resistance in Parallel Channels* by O. Reynolds, *Philosophical Transactions, Royal Society*, London, 1883.

of increasing velocity. Figure 47a, with the lowest velocity, definitely shows laminar flow, whereas Figures 47d and 47e show turbulent flow.

52. Critical Reynolds' numbers

The dimensionless Reynolds' number $N_R = \rho VD/\mu$ can be used as a criterion for determining whether the pipe flow with *any* fluid is laminar or turbulent. V is the average velocity of the fluid in the pipe and D is the internal pipe diameter. A relatively small N_R indicates that viscous forces predominate (as in laminar flow), whereas a relatively large N_R indicates that inertia forces predominate (as in turbulent flow).

Experiments show that the stable form of motion for pipe flow is normally laminar for N_R less than 2000. Below $N_R = 2000$ an initially turbulent flow, with its typical irregular mixing or eddying motion, cannot be maintained indefinitely. Thus a *lower critical velocity* V_c may be established in which

$$V_c = 2000\left(\frac{\mu}{\rho D}\right). \tag{38}$$

If the actual average velocity V is below V_c, the stable flow will be laminar.

For usual conditions the pipe flow is turbulent for values of N_R above about 3000. In the transition range $N_R = 2000$ to $N_R = 3000$ there are various possible conditions, depending upon the initial disturbances, the pipe entrance, and the pipe roughness. By very careful manipulation of the apparatus, laminar flow has been obtained with values of N_R considerably above 3000; such flow, however, is not inherently stable, for any disturbance, once started, tends to break up the laminar pattern. Laminar flow at values of N_R considerably above 3000 is a phenomenon which is somewhat similar to supersaturation or to undercooling. Normally, laminar flow does not occur for values of N_R above about 3000.

It is to be noted that the foregoing critical values of Reynolds' number apply only to pipe flow. Critical values of Reynolds' numbers for other cases, as for flow around various bodies immersed in a stream, are different from those given in the preceding discussion. Further, it is difficult to specify exactly one universal critical Reynolds' number, or one specific value which will provide a very sharp and general distinction for all pipe installations. Turbulent flow depends upon the pipe roughness, initial disturbances, entrance conditions, and other factors. Hence, the several critical values given in the foregoing discussion have been qualified.

53. Energy relations for pipe flow

Using the notation shown in Fig. 48, the case is taken in which the fluid is incompressible and there is no pump, turbine, or similar machine between sections 1 and 2. The general energy equation then becomes

$$(q + u_1 - u_2)778 = \left(\frac{p_2 - p_1}{w}\right) + \left(\frac{V_2^2 - V_1^2}{2g}\right) + (z_2 - z_1). \quad (39)$$

As the fluid flows through the pipe, some mechanical energy is degraded into unavailable energy; there is a friction loss, or a *head loss* due to

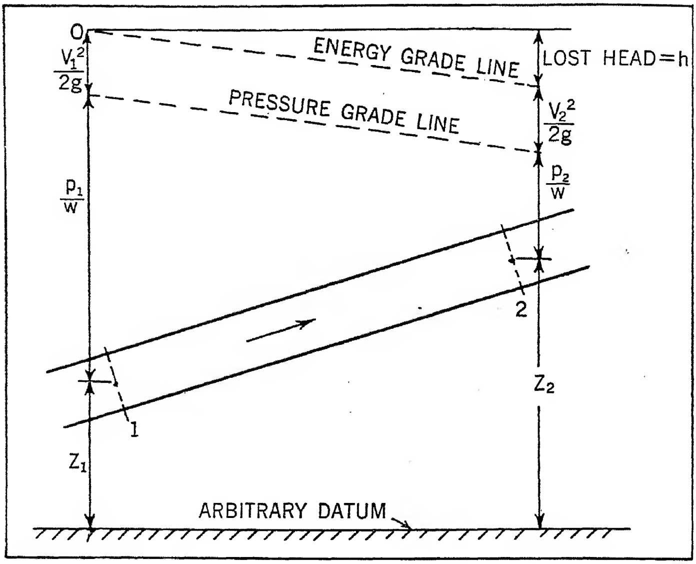

FIG. 48. Notation for pipe flow.

viscosity and turbulence. Let h represent this lost head. Then the general energy equation can be written as

$$-h = \left(\frac{p_2 - p_1}{w}\right) + \left(\frac{V_2^2 - V_1^2}{2g}\right) + (z_2 - z_1)$$

or

$$h = \left(\frac{p_1 - p_2}{w}\right) + \left(\frac{V_1^2 - V_2^2}{2g}\right) + (z_1 - z_2). \quad (40)$$

Each term in Equation (40) is expressed in units of mechanical energy per unit weight of fluid flowing. The lost energy h, for example, can be stated in terms of foot-pounds per pound of fluid, or simply feet, or some other net unit of length.

The terms of the preceding energy equation can be represented by vertical ordinates, as is done in Fig. 48. The *pressure grade line* is an imaginary line giving the sum of $\left(z + \frac{p}{w}\right)$ at any point along the pipe;

it is the line which a series of manometers or piezometers would indicate. Plotting the *total head* $\left(z + \frac{p}{w} + \frac{V^2}{2g}\right)$ at each point gives the *energy grade line*. The energy grade line slopes down from the horizontal in the direction of flow. The vertical distance between the energy grade and a horizontal line through 0 represents the lost head. If the area at section 1 equals the area at section 2, the equation of continuity specifies that $V_1 = V_2$. In this case the pressure grade line is parallel to the energy grade line, or

$$h = \left(\frac{p_1}{w} + z_1\right) - \left(\frac{p_2}{w} + z_2\right). \tag{41}$$

If the pipe is horizontal, then $z_2 = z_1$, and the head loss h equals $\frac{p_1 - p_2}{w}$.

54. Friction losses in circular pipes

A great deal of work has been done on the determination of friction losses in circular pipes. It is customary to express the lost head h in the form

$$h = f\left(\frac{l}{D}\right)\frac{V^2}{2g}, \tag{42}$$

where D is the internal pipe diameter, l is the pipe length, and f is a dimensionless friction coefficient. Dimensional analysis and dynamic similarity show that f is a function of Reynolds' number, and experimental data has been correlated on this basis. Figure 49 shows a plot of f as a function of N_R based upon a large number of different tests. Both f and N_R are dimensionless; any set of consistent units can be used in computing Reynolds' number, and in using Equation (42). It has been shown that the friction factor is independent of the fluid flowing for the same values of N_R.

For laminar flow, the friction factor is independent of the pipe surface or roughness. For turbulent flow, however, the friction coefficient is definitely a function of the character of the wall surface or roughness. In some cases the definition, specification, and estimation of pipe roughness present real difficulties. The general problem of roughness has not yet been completely solved. The organization of data given in Fig. 49, which is due to Pigott, seems to be the best available at the present time for practical problems.

One common important type of problem is the determination of pressure drop for a given length, diameter, dynamic viscosity, density, and average velocity (or rate of discharge). In solving such a problem the first step is to calculate N_R. Reference to the particular curve in

CURVES NO. 1 AND 2—SMOOTH DRAWN TUBES

Curve No.	Clean steel, wrought iron	Clean galvanized	Best cast iron, cement, light riveted sheet ducts	Average cast iron, rough-formed concrete	First-class brick, heavy riveted
3	14–42″ dia.	30″ dia.	48–96″ dia.	96″ dia.	220″ dia.
4	6–12″	10–24″	20–48″	42–96″	84–204″
5	4–5″	6–8″	12–16″	24–36″	48–72″
6	2–3″	3–5″	5–10″	10–20″	20–42″
7	$1\frac{1}{2}$″	$2\frac{1}{2}$″	3–4″	6–8″	16–18″
8	1–$1\frac{1}{4}$″	$1\frac{1}{2}$–2″	2–$2\frac{1}{2}$″	4–5″	10–14″
9	$\frac{3}{4}$″	$1\frac{1}{4}$″	$1\frac{1}{2}$″	3″	8″

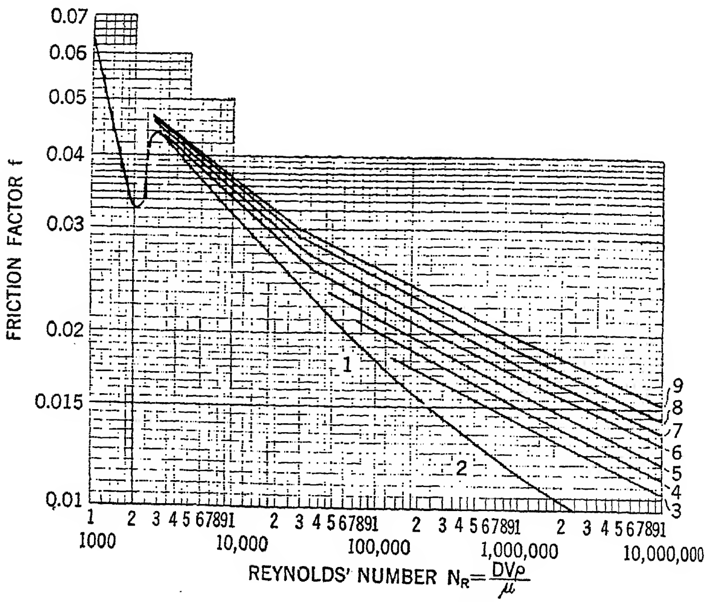

FIG. 49. Friction factors for flow in circular pipes.[2]

[2] Data for curve No. 1 after Stanton and Pannell, data for curve No. 2 after Nikuradse, and data for curves No. 3 to 9 after Pigott. See the first three references listed at end of this chapter, page 87.

Fig. 49, selected with the aid of the table, gives the friction coefficient. Use of Equation (42) gives the head loss.

EXAMPLE. Benzene at 50° Fahrenheit (specific gravity = 0.90) flows through a horizontal smooth drawn tube 2 inches in diameter with an average velocity of 2.9 feet per second. Taking the dynamic viscosity of benzene as 1.6×10^{-5} slugs per foot-second (see Fig. 42), calculate the pressure drop in 600 feet of pipe.

SOLUTION.

$$N_R = \frac{0.90(62.4)2(2.9)}{32.2(12)1.6(10^{-5})} = 5.27 \times 10^4.$$

This flow is turbulent. From Fig. 49, $f = 0.021$. Then the lost head is

$$h = 0.021\left(\frac{600}{\frac{2}{12}}\right)\frac{(2.9)^2}{2(32.2)} = 9.87 \text{ feet (or foot-pounds per pound).}$$

The pressure drop Δp is

$$\Delta p = wh = 0.90(62.4)\frac{9.87}{144} = 3.85 \text{ pounds per square inch.}$$

Subsequent chapters, such as those dealing with flow meters and with the resistance of moving immersed bodies, will bring out the fact that other flow coefficients are correlated with Reynolds' number, and that the method of calculation in many of these other problems is somewhat similar to that used in pipe flow problems.

55. Hagen-Poiseuille law for laminar flow in circular pipes

Careful experiments have shown that for fully developed *laminar* flow in all circular pipes $f = 64/N_R$. For this case the pressure drop in a horizontal pipe becomes

$$p_1 - p_2 = \Delta p = wh = w\frac{64}{N_R}\left(\frac{l}{D}\right)\frac{V^2}{2g} = \frac{32\mu l V}{D^2}. \tag{43}$$

The rate of discharge $Q = \dfrac{\pi D^2}{4}V$. Then

$$\Delta p = \frac{128\mu l Q}{\pi D^4}. \tag{44}$$

Equations (43) and (44) are forms of the so-called Hagen-Poiseuille law (after G. Hagen and L. J. M. Poiseuille). This law can be derived on an analytical basis, as will be done later, to show that the mathematical analysis of laminar flow checks very accurately with experimental results.

In Fig. 50 the pressure difference $p_1 - p_2$ exerts the force $(p_1 - p_2)\pi y^2$ on a cylindrical mass of fluid of radius y. The opposing force consists of friction on the curved cylindrical surface; this total force is $2\pi y l \tau$.

Equating these two forces, for steady, non-accelerated flow, gives the relation for shear stress:

$$\tau = \left(\frac{p_1 - p_2}{l}\right)\frac{y}{2} = \left(\frac{\Delta p}{l}\right)\frac{y}{2}. \tag{45}$$

For laminar flow, $\tau = \mu(du/dy)$. With the coordinate system employed and the highest velocity at the pipe center, du is a decrement. Then

$$\frac{du}{dy} = -\frac{\Delta p}{2\mu l}y.$$

Integrating, and determining the integration constant by the fact that $u = 0$ at $y = D/2$ (fluid adheres to the pipe wall), there results

$$u = \frac{\Delta p}{4\mu l}\left[\frac{D^2}{4} - y^2\right]. \tag{46}$$

Equation (46) indicates a parabolic velocity distribution, as represented in Fig. 50. The maximum velocity occurs at the center where

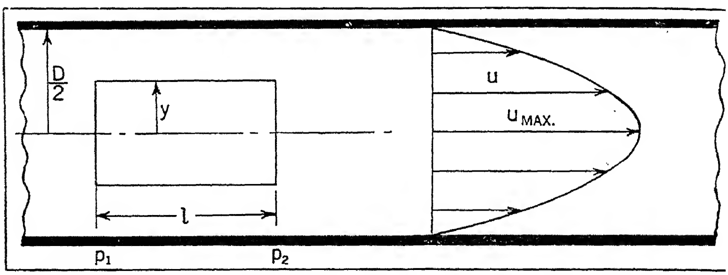

FIG. 50. Laminar flow in a horizontal circular pipe, with a parabolic velocity distribution.

$y = 0$. Thus

$$u_{\text{max.}} = \frac{\Delta p D^2}{16\mu l}. \tag{47}$$

Q can be determined by integration

$$Q = \int_0^{D/2} 2\pi y\,dy\,u = \frac{\pi}{\mu l}\Delta p\frac{D^4}{128}, \tag{48}$$

which checks Equation (44). The Hagen-Poiseuille law provides the best way for determining the dynamic viscosity of a fluid. Inspection of Equations (43) and (47) shows that $V = u_{\text{max.}}/2$, that the average velocity is one-half the maximum velocity at the center. It is to be emphasized that the Hagen-Poiseuille law applies only for *laminar* flow in circular pipes. Before judging the results of a calculation based on the

Hagen-Poiseuille law, the type of flow should be established, as by a calculation of the Reynolds' number.

Equation (45) indicates that the shear-stress distribution across a pipe section for laminar flow is a linear function of the radial distance from the pipe center. Equation (43) shows that the pressure drop for laminar flow is directly proportional to the first power of the velocity. For turbulent flow, the pressure drop is directly proportional to some power of the velocity between 1 and 2. The value of the power can be calculated by referring to the experimental data given in Fig. 49.

56. Velocity distributions in circular pipes

The velocity distribution at a pipe section depends upon the length of pipe preceding the section. At the rounded entrance in Fig. 51 the

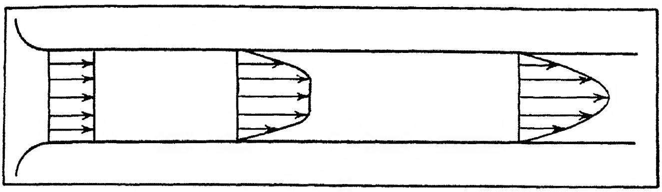

FIG. 51. Velocity profiles along a pipe for laminar flow.

velocity profile is approximately uniform except for a thin film at the walls. For laminar flow, a short distance downstream the velocity profile becomes a combination of a nearly parabolic distribution at the walls and a core in which the velocity is nearly uniform. Fully developed laminar flow, with the parabolic velocity profile, is reached further downstream. The distance for complete transition depends upon N_R.

For fully developed flow, the velocity profile becomes flatter as N_R increases. Figure 52 shows a comparison of three velocity profiles at different values of N_R, but each for the same average velocity. The ratio $V/u_{max.}$ is sometimes called the *pipe coefficient* or *pipe factor*, where $u_{max.}$ is the velocity at the center of the pipe. Figure 53 shows a plot of $V/u_{max.}$ against N_R, based upon data presented by Stanton and Pannell.[3] For laminar flow the ratio is 0.50, whereas for turbulent flow the ratio is greater than 0.50.

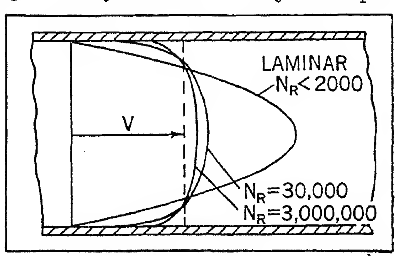

FIG. 52. Comparison of velocity profiles in a smooth pipe for the same average velocity V and different Reynolds' numbers.

[3] See the first reference at the end of this chapter.

Kármán and Prandtl have shown that velocity distributions for turbulent flow may be expressed by power laws over a wide range. Let u be the velocity at a distance y from the wall, and let r be the pipe radius.

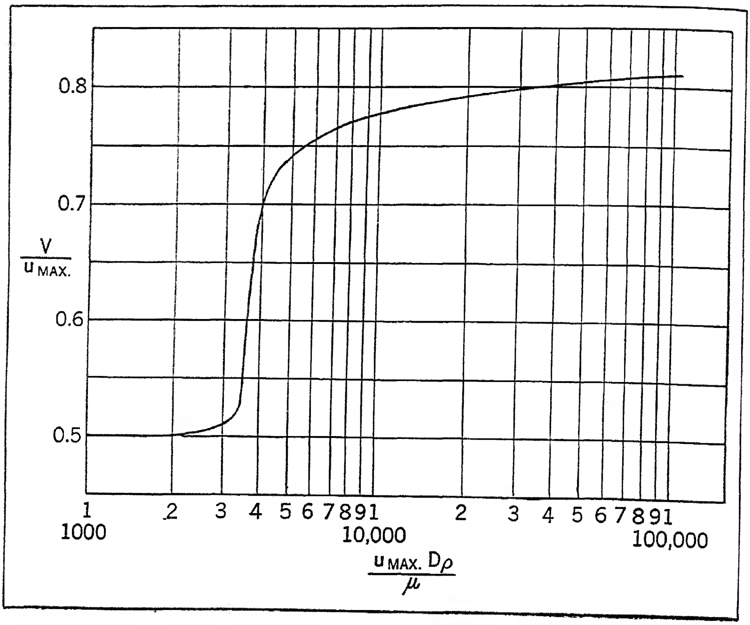

FIG. 53. Pipe factors as a function of Reynolds' number; $u_{\max.}$ is the velocity at the center of the pipe.

Then the velocity distribution in the main stream can be represented by the relation

$$u = u_{\max.}\left(\frac{y}{r}\right)^n. \tag{49}$$

The value of n is $\frac{1}{7}$ for turbulent flow in smooth tubes up to values of N_R of about 100,000. If the pipe factor (Fig. 53) is taken as 0.82, then $u_{\max.} = 1.22V$. For this case,

$$u = 1.22V\left(\frac{y}{r}\right)^{1/7}, \tag{50}$$

which relation agrees fairly well with experimental results. For higher values of N_R, above about 100,000 to about 400,000, the velocity distribution for smooth tubes is better represented with $n = \frac{1}{8}$. Higher powers, such as $n = \frac{1}{5}$, are found with rough pipes.

Differentiation of Equation (50) shows that the velocity gradient du/dy becomes infinitely large as y approaches zero. This result would indicate that the shear stress becomes infinitely large as y approaches zero. The impossibility can be explained by noting that the seventh-root relation for turbulent velocity distribution ceases to be valid at the pipe wall. The seventh-root relation only holds for the main stream up to a very small distance from the wall. In this short distance from the wall there is a thin layer in which the flow is laminar; sometimes this thin layer is called a laminar sublayer. The velocity gradient for laminar flow becomes finite at the wall, as can be seen by examining the parabolic velocity distribution for such flow.

57. Turbulent flow in circular pipes

The exact nature or mechanism of turbulent flow has not been completely established at the present time. For extensive details the reader is referred to the outstanding publications of von Kármán, Dryden, Bakhmeteff, and others (see references at end of this chapter). A few brief remarks on shear stress, however, might be helpful in comparing laminar flow with turbulent flow.

In Article 39, Chapter 5, the dynamic viscosity μ was defined for laminar flow as the coefficient in the following expression for shear stress:

$$\tau = \mu \frac{du}{dy}.$$

For turbulent flow the shear stress can be written as

$$\tau = (\mu + \epsilon) \frac{du}{dy}. \tag{51}$$

where ϵ (Greek letter epsilon) has the dimensions of dynamic viscosity. ϵ has been called the *exchange coefficient, mechanical viscosity,* or *eddy viscosity.* ϵ is not a physical characteristic constant of the fluid, like μ, but depends upon the Reynolds' number and other parameters.

Murphree[4] has calculated values of the ratio between eddy viscosity in the main stream and dynamic viscosity for the flow through some circular pipes. His results show that the eddy viscosity increases with Reynolds' number. For example, for $N_R = 3000$ the eddy viscosity is 7.2 times the dynamic viscosity, whereas for $N_R = 500{,}000$ the eddy viscosity is 470 times the dynamic viscosity. The shear stress in turbulent flow may be considerably higher than that for laminar flow.

[4] *Relation between Heat Transfer and Fluid Friction* by E. V. Murphree, *Industrial and Engineering Chemistry*, vol. 24, July, 1932, page 726.

58. Direct computation of pipe-line flow

There are various practical pipe flow problems which cannot be solved directly by Fig. 49 alone. For example, there are the following cases:

(a) Given ρ, μ, D, and the pressure drop Δp in a length l, what is the velocity V or the discharge Q?

(b) Given ρ, μ, Q, Δp, and l, what is D?

In either case the Reynolds' number cannot be calculated; in one case the velocity is not known; in the other case the diameter is not known. At first thought it might appear that a trial-and-error method is necessary. S. P. Johnson,[5] however, has clearly indicated a direct method of solution for such problems if the relative wall roughness is known; a trial-and-error method is not necessary. He arrived at this result by means of dimensional analysis. The practical usefulness of the methods suggested by Johnson is great.

For pipes of the same relative roughness, there is a definite functional relation between the variables ρ, μ, D, Δp, l, and V (or Q). Viewing the general problem of pipe flow, Reynolds' number is only one convenient dimensionless ratio; there are other dimensionless ratios which would be helpful in solving cases (a) and (b). Specifically, for case (a) one ratio should not include the velocity; for case (b) one ratio should not include the diameter. Suitable dimensionless ratios are the following; one will be called an S number and the other will be called a T number:

$$S = \frac{[p'D^3\rho]^{1/2}}{\mu}, \qquad T = \frac{[Q^3p'\rho^4]^{1/5}}{\mu}.$$

where p' is the pressure gradient $\Delta p/l$. S does not include V, and T does not include D. Any set of consistent units could be used for each ratio. For example, the following set could be employed: Δp in pounds per square foot, l in feet, μ in slugs per foot-second, ρ in slugs per cubic foot, Q in cubic feet per second, D in feet, and V in feet per second.

The diagram of Fig. 54 is the result of adding the S and T coordinates to the diagram shown in Fig. 49. For case (a) the procedure is to compute the S number and follow the S coordinate to the pipe curve applying. Then f or N_R is determined, from which V and Q can be calculated. For case (b) no difficulties are presented if the flow proves to be laminar. Then the friction loss is independent of the pipe roughness; there is only one curve for the relation between f and N_R. The procedure is to compute the T number and follow the T coordinate to the pipe curve applying. Thus f or N_R is determined, from which D can be calculated. This

[5] *A Survey of Flow Calculation Methods* by S. P. Johnson, *Preprinted Papers and Program, A.S.M.E. Summer Meeting*, June, 1934, page 98.

CURVES NO. 1 AND 2—SMOOTH DRAWN TUBES

Curve No.	Clean steel, wrought iron	Clean galvanized	Best cast iron, cement, light riveted sheet ducts	Average cast iron, rough-formed concrete	First-class brick, heavy riveted
3	14–42″ dia.	30″ dia.	48–96″ dia.	96″ dia.	220″ dia.
4	6–12″	10–24″	20–48″	42–96″	84–204″
5	4–5″	6–8″	12–16″	24–36″	48–72″
6	2–3″	3–5″	5–10″	10–20″	20–42″
7	$1\frac{1}{2}$″	$2\frac{1}{2}$″	3–4″	6–8″	16–18″
8	1–$1\frac{1}{4}$″	$1\frac{1}{2}$–2″	2–$2\frac{1}{2}$″	4–5″	10–14″
9	$\frac{3}{4}$″	$1\frac{1}{4}$″	$1\frac{1}{2}$″	3″	8″

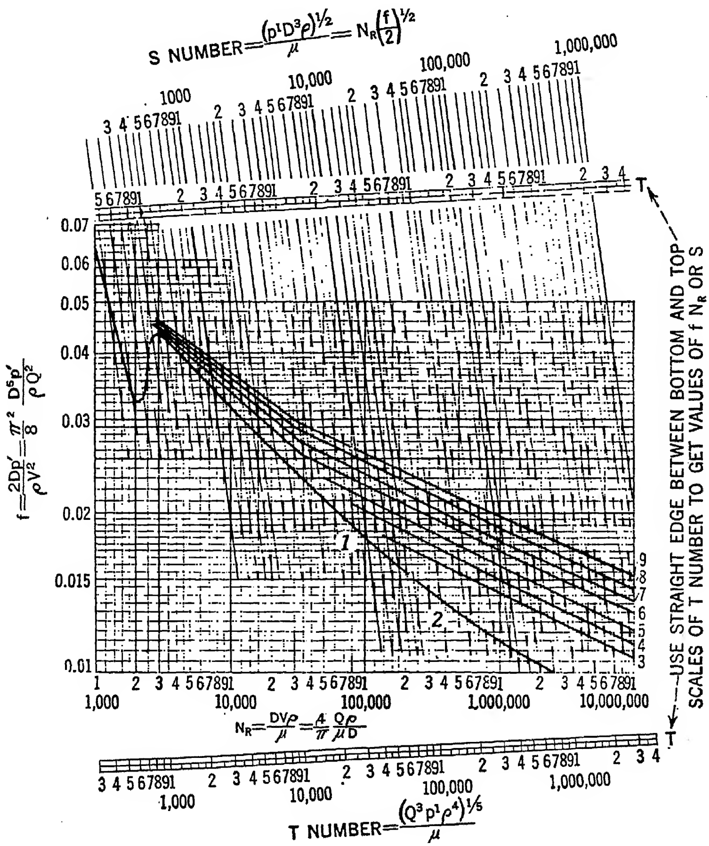

FIG. 54. Relation between friction factor and Reynolds' number for flow in circular pipes, with additional coordinates S and T.

same procedure can be applied if the flow proves turbulent, provided the relative roughness, or the curve to be used, is known. The relative roughness may not be known exactly, and hence some trial may be necessary. The T coordinate, however, reduces the amount of trial and error which would exist if a diagram like Fig. 49 were used alone.

The friction coefficient f in Fig. 54 is written as

$$f = \frac{2Dp'}{\rho V^2} = \frac{\pi^2}{8}\left(\frac{D^5 p'}{\rho Q^2}\right).$$

There is another type of problem in which the Reynolds' number cannot be calculated in the first step. In some problems, as those involving the determination of the temperature (and consequently viscosity) at which it is economical to pump an oil a long distance, D, p', ρ, V (or Q) are given and it is required to determine an allowable value of the dynamic viscosity. For such a problem f can be calculated from one of the preceding forms. Following the horizontal value of f to the curve applying gives the N_R, from which the dynamic viscosity, and hence the temperature, can be determined.

59. Minor losses in pipe lines

Energy losses in pipe lines due to changes in sections, bends, fittings, and valves are commonly termed *minor losses.* Such losses are in addition to the usual pipe friction loss. Minor losses may be small for a long pipe line, but may be appreciable for a short line.

A minor loss head h_0 is usually expressed as equal to $K(V^2/2g)$, where K is a dimensionless coefficient. Reliable values of minor loss coefficients for all fluids have not been fully established at the present time; real difficulties have been encountered in trying to correlate experimental data. The values given in this particular article are to be regarded as approximations, because they are based on somewhat limited experimental results. Special references dealing with the particular loss in question should be consulted if greater precision is desired.

Fig. 55 gives the minor loss for a sudden enlargement. Table 5

TABLE 5
VALUES OF K_1 FOR A SUDDEN CONTRACTION

$\frac{D_2}{D_1}$	0.1	0.3	0.5	0.7	0.9
K_1	0.45	0.39	0.33	0.22	0.06

gives values of K_1 for the sudden contraction shown in Fig. 55. Values of minor losses for pipe entrances are shown in Fig. 56.

The head loss in pipe fittings and valves may also be expressed as equal to $K_2(V^2/2g)$, where V is the velocity in a pipe of the nominal

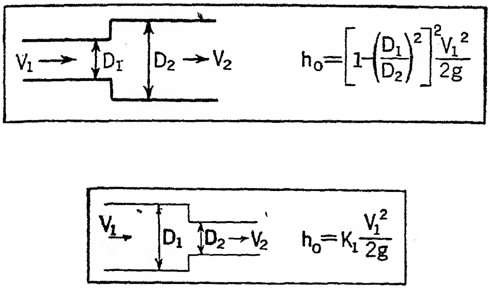

FIG. 55. (*Top*) Sudden enlargement; (*bottom*) sudden contraction.

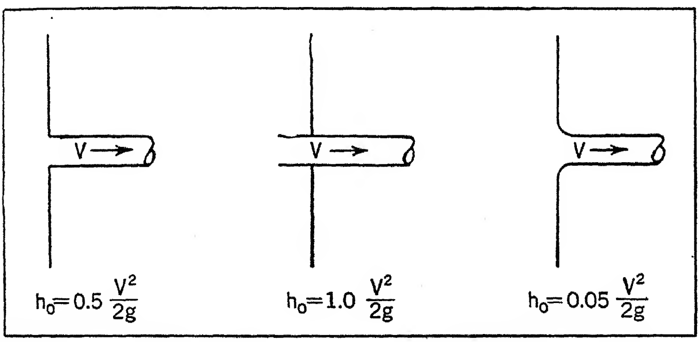

FIG. 56. Pipe entrance losses.

size of the fitting. Some values of K_2, as compiled by Crane Co.,[6] are listed in Table 6.

60. Friction losses in noncircular pipes

The foregoing discussions in this chapter have been confined to the flow of fluids in pipes of circular cross section. In some applications it is necessary to make flow calculations for noncircular and annular shapes.

[6] *Flow of Fluids Through Valves, Fittings, and Pipe,* Technical Paper No. 409, Crane Co., May, 1942, page 20.

TABLE 6

MINOR LOSS COEFFICIENTS FOR VALVES AND FITTINGS

Valve or Fitting	K_2
Globe valve, wide open	10
Angle valve, wide open	5
Gate valve, wide open	0.19
Gate valve, $\frac{1}{4}$ closed	1.15
Gate valve, $\frac{1}{2}$ closed	5.6
Gate valve, $\frac{3}{4}$ closed	24.0
Return bend	2.2
Standard tee	1.8
Standard elbow	0.9
Medium sweep elbow	0.75
Long sweep elbow	0.60
45-degree elbow	0.42

A parameter called the *profile radius, hydraulic mean depth,* or *hydraulic radius* has been devised to handle these problems. This term is defined as

$$\text{profile radius} = R = \frac{\text{cross-sectional area}}{\text{wetted perimeter}}.$$

For a circular pipe flowing full, the profile radius R equals $D/4$. If $4R$ is substituted for D, Reynolds' number becomes

$$N_R = \frac{4VR\rho}{\mu}, \tag{52}$$

and Equation (42) can be written as

$$h = \frac{f}{4}\left(\frac{l}{R}\right)\frac{V^2}{2g}. \tag{53}$$

Use of Equations (52) and (53) in connection with Fig. 49 for ordinary shapes gives results which are in fair agreement with experimental data if the flow is turbulent. Use of the profile radius for laminar flow may give very inaccurate values. Lamb[7] presents some theoretical results for laminar flow in sections other than circular.

61. Secondary flow in bends

Under some circumstances, as in the flow around a bend in a pipe, there may be a *secondary flow* superimposed upon the primary or main flow. The additional energy dissipation resulting from the secondary flow may be undesirable, or the secondary flow may affect the operation of a machine or meter. This secondary flow, on the other hand, may be useful in separating particles, like silt, carried by the fluid. Before discussing this phenomenon, it will be helpful to bring out some general features regarding the flow in a curved path.

[7] *Hydrodynamics* by H. Lamb, Sixth Edition, Cambridge University Press, 1932, page 586.

Consider the flow between two concentric streamlines an infinitesimal distance apart, as represented in Fig. 57. The radius of curvature of the path is r, and the tangential, linear velocity is V. The infinitesimal element has a height dr, and an area dA along the curved surface. The mass of this element is $\rho dr dA$. The normal or radial acceleration is V^2/r. The centrifugal force acting on the element has the magnitude $\rho dA dr(V^2/r)$. The pressure varies from p to $p + dp$ as the radius varies from r to $r + dr$. The centrifugal force on the fluid element is just balanced by the resultant force due to the pressures over the surfaces. If

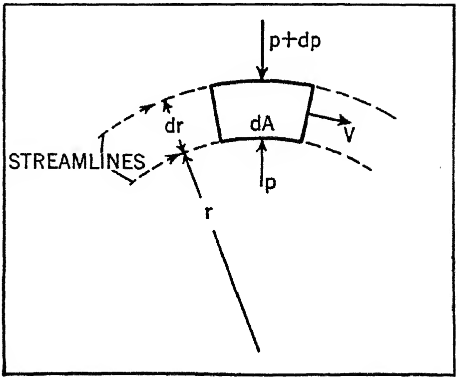

FIG. 57. Flow in a curved path.

infinitesimals of higher order than the first are neglected, a force balance in the radial direction gives

$$dp dA = \rho dr dA \frac{V^2}{r}, \qquad dp = \rho dr \frac{V^2}{r}. \tag{54}$$

The pressure increases with radius in curved flow. There is a fall in pressure per unit radial distance toward the center of curvature by the amount $\rho(V^2/r)$; the pressure gradient $dp/dr = \rho(V^2/r)$.

If, for example, fluid is circulating in a stationary flat-bottom cylindrical vessel, as shown in Fig. 58, the primary pattern can be indicated by concentric circles (shown dotted in the figure). The velocity in a layer next to the bottom of the vessel, as AA, is retarded by the bottom; the layer near the bottom moves at a lower velocity than some layer above it, as BB. The pressure, however, of one layer is transmitted to the layer below it; the pressure gradient dp/dr in each adjacent layer is the same, namely $\rho V^2/r$. In a layer near the bottom the radius of curvature is smaller than that in the layer above it, because the velocity in the lower layer is smaller. Besides a concentric primary flow, there is

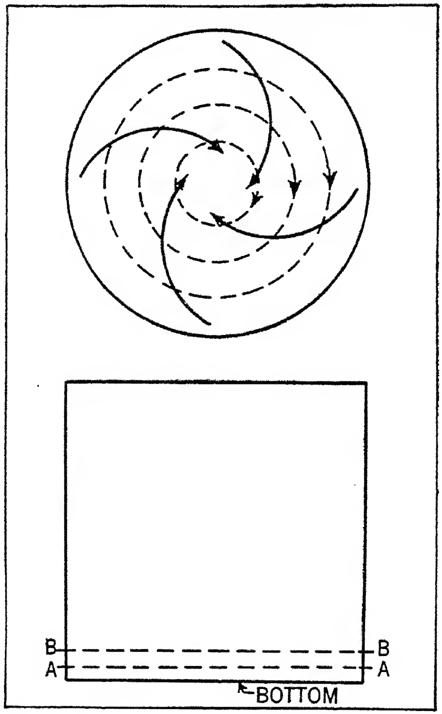

Fig. 58. Secondary flow in a flat-bottom cylindrical vessel.

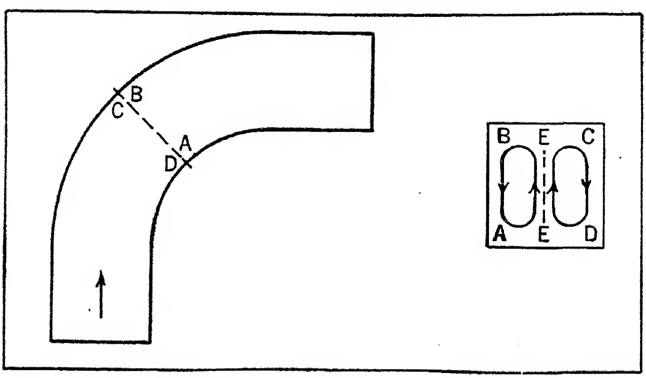

Fig. 59. Secondary flow in a pipe bend.

thus superimposed a secondary flow; the resultant flow follows an inward spiral path.

Observation shows that small particles carried by such a stream are heaped toward the center of the bottom. In natural rivers and streams the secondary flow at bends tends to pile up sand and gravel at the inner side of the bend, and to deepen the bend at the outer side. Secondary flow thus tends to make the bend more pronounced or more meandering.

Figure 59 illustrates diagrammatically the secondary flow observed in some pipe bends. The side AD is at the inner radius, whereas the side BC is at the outer radius. Viewing one side only, for example, the velocity in the layers close to the wall AB is less than that in the central plane EE. The pressure gradient, however, is transmitted from one layer to the next. Thus there is a tendency for the secondary flow to proceed from B to A. The rest of the fluid, by continuity, is forced to stream, slowly, to complete the circuit. Superimposed upon the primary flow is the secondary flow consisting of two spirals.

The secondary flow in a bend may be reduced or eliminated by suitable guide vanes. Also, since the walls AB and CD play such a prominent rôle in developing the secondary flow, the secondary flow may be reduced by making the sides AB and DC small in comparison with BC and AD. Two-dimensional flow in a curved channel may be realized with such a deep, narrow channel.

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The Mechanics of Turbulent Flow by B. A. Bakhmeteff, Princeton University Press, 1936.

PROBLEMS

67. 740 gallons of gasoline (specific gravity = 0.75) per minute flow through a horizontal clean steel pipe 8 inches in diameter and 8 miles long. Kinematic viscosity is 0.01 centimeter squared per second. What is the pressure drop?

68. Air flows through a clean galvanized pipe 3 inches in diameter. At one section the pressure is 150 pounds per square inch gage, the temperature is 90° Fahrenheit, and the average velocity is 45 feet per second. What is the pressure drop in 200 feet of pipe if the density change is negligible?

69. 1500 gallons of benzene at 50° Fahrenheit (specific gravity = 0.90) are to be pumped per hour through a $1\frac{1}{4}$-inch standard wrought iron pipe (1.38 inches actual inside diameter). The total length of horizontal pipe is 1000 feet. The overall efficiency of the pump is 60 per cent. Find the horsepower input to the pump.

70. Castor oil at 59° Fahrenheit flows through a horizontal wrought-iron pipe 1 inch in diameter and 5 feet long at a rate of 0.090 pound per second. What is the pressure drop?

71. Water at 59° Fahrenheit flows through a 6-inch diameter average cast-iron pipe with an average velocity of 4.0 feet per second. What is the pressure drop in a length of 120 feet?

72. Calculate the power which must be added to the fluid to pump 22 cubic feet per minute of ethylene bromide at 20° centigrade through a horizontal smooth-drawn tube 900 feet long and 2 inches in diameter. The dynamic viscosity is 3.60×10^{-5} slugs per foot-second; the specific gravity is 2.17.

73. Walker, Lewis, and McAdams, in their book, *Principles of Chemical Engineering,* use a Reynolds' number involving the following terms: diameter in inches, velocity in feet per second, specific gravity, and viscosity in centipoises. Determine the factor by which this number must be multiplied in order to give the Reynolds' number in any consistent units.

74. 117.9 cubic feet of water per minute flows through a smooth horizontal drawn tube. The pressure drop in 1000 feet is 18.2 pounds per square inch. What is the pipe diameter?

75. Air flows through a clean galvanized horizontal pipe 12 inches in diameter. The pressure drop in 100 feet is 0.040 pound per square inch. For standard conditions, what is the rate of discharge?

CHAPTER 8

Fluid-measuring Instruments

> I have seen so much of the danger arising from presenting results or rules involving variable coefficients in the form of algebraic formulas which the hurried or careless worker may use far beyond the limit of the experimental determination that I present the results mainly in the form of plotted curves which cannot be thus misused and which clearly show the margin of uncertainty and the limitations of the data.—John R. Freeman.[1]

Problems concerning the use of instruments for fluid measurement and control are encountered frequently in industrial and research work. The present chapter covers only a few of the more common devices which are not covered in other parts of this book. Further details about installation and operation can be found in the specialized reference literature, as, for example, in the excellent reports on fluid meters published by the American Society of Mechanical Engineers. The quantitative relations given in this chapter are for incompressible flow. Compressible flow is discussed in Chapter 12.

Force, length, and time are quantities which can be measured very accurately. It is good technique to refer all derived measurements, such as pressure, velocity, and rate of discharge, as directly as possible to the primary standards of force, length, and time. For example, in order to determine the rate of discharge from a pump or fluid machine, good accuracy can be obtained by weighing the discharge during a measured time interval. In some cases, however, economy and convenience may dictate the use of a secondary instrument whose accuracy depends solely on a calibration or assumed coefficients.

62. Pitot tubes

Various types of pitot (named after Henri Pitot) tubes are frequently used for measuring velocity. As shown in Fig. 60, V_0 is the velocity some distance ahead of the simple pitot tube. The pressure p_0 in the undisturbed fluid is commonly called the *static* pressure. Static pressure is the force per unit area exerted by a fluid on a surface at rest relative to the fluid. The streaming fluid is brought to rest at the nose or stagnation point of the instrument. The fluid at rest in the pitot tube can be connected to some manometer or pressure gage, to give a measurement of the

[1] *Experiments upon the Flow of Water in Pipes and Pipe Fittings,* A.S.M.E., 1941.

total or *stagnation pressure* p_s. Applying the energy equation for incompressible frictionless flow between point O and the nose gives

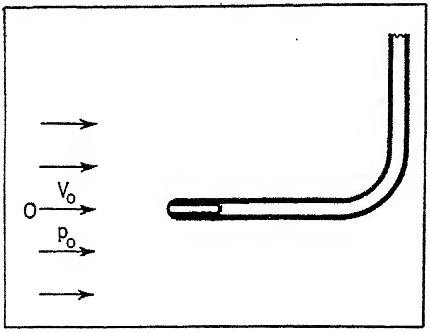


FIG. 60. Simple pitot tube.

$$\frac{p_0}{w} + \frac{V_0^2}{2g} = \frac{p_s}{w}, \qquad (55)$$

or

$$V_0 = \sqrt{\frac{2(p_s - p_0)}{\rho}}. \qquad (56)$$

Note that the specific weight w in Equation (55) is the specific weight of the fluid flowing. The pitot may be connected to a manometer containing a fluid which is either the same as that or different from that of the flowing stream. The term p_s/w is sometimes called the *total-pressure head*, the term $V_0^2/2g$ the *velocity head*, and the term p_0/w the *static-pressure head*. If the difference in pressure $p_s - p_0$ can be measured, the velocity V_0 can be computed by Equation (56).

In general, it is not difficult to measure the total pressure p_s accurately. On the other hand, it is sometimes difficult to obtain an accurate measurement of the static pressure or the difference $p_s - p_0$ in a stream. If the pitot tube is inserted in a pipe, p_0 may be measured by means of a static-pressure opening in the pipe wall, provided the static-pressure distribution across the pipe is constant.

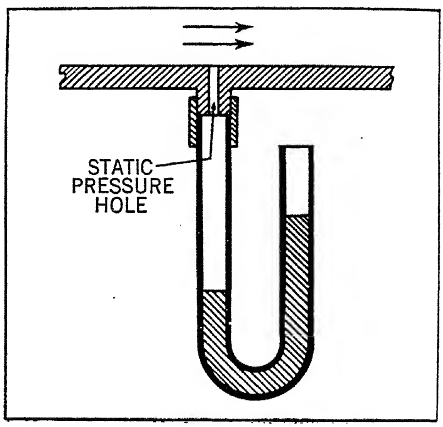


FIG. 61. Illustration of a static-pressure measurement by means of a manometer.

Figure 61 illustrates how the static pressure at a wall can be measured. Any type of pressure-measuring instrument could be employed. The fluid flows past the static-pressure hole, opening, or *tap* but remains at rest in the hole itself if its dimensions are small enough.

Static-pressure holes might be included in a velocity-measuring instrument, as in the combined pitot tube shown in Fig. 62. A differential gage across the static-pressure and total-pressure connections will

give the dynamic pressure $\frac{1}{2}\rho V_0^2 = p_s - p_0$ directly. A pitot tube should be calibrated for precise determinations. For average engineering work, Equation (56) is often used directly without any modification.

One type of pitot tube, sometimes called a *direction-finding tube,*[2] is convenient and useful for measurements of velocity in both direction and magnitude. This instrument can be explained by referring to the two-dimensional flow around a cylinder whose axis is perpendicular to the flow some distance ahead of the cylinder. Let p be the pressure at any point on the surface of the cylinder. Experiments show that the distribution of the pressure difference $(p - p_0)$ is somewhat as represented by

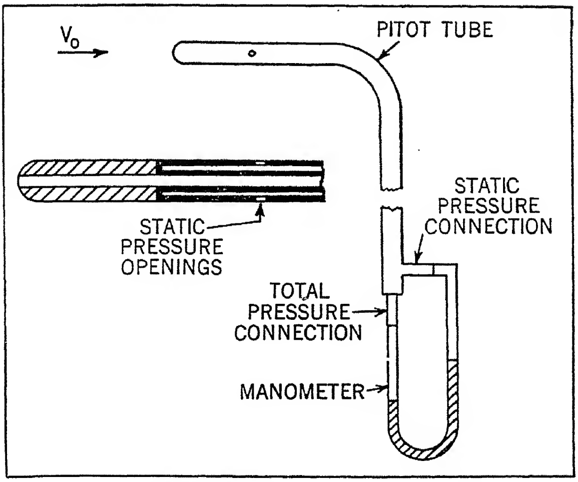


Fig. 62. Combined pitot tube.

the radial ordinates in Fig. 63. At the stagnation point A, the pressure difference $p - p_0 = \frac{1}{2}\rho V_0^2$. The pressure difference decreases for successive points from A to B (or C). At points B and C, $p - p_0 = 0$; if there were an opening in the surface of the cylinder at the *critical angle,* the pressure transmitted to a gage would be truly static. Experiments show that this critical angle is $39\frac{1}{4}°$ for an average range of turbulent flow.

One possible construction of the direction-finding pitot is shown in Fig. 64; the pitot consists of a cylindrical tube, with two holes and compartments, whose axis is perpendicular to the stream. In the position shown in Fig. 64*a*, a pressure reading from either compartment will give

[2] *Experimental Determinations of the Flow Characteristics in the Volutes of Centrifugal Pumps* by R. C. Binder and R. T. Knapp, *A.S.M.E. Transactions,* November, 1936, page 649.

p_0. If the tube is rotated about its axis so that one opening is in line with V_0, this opening will give p_s. The velocity can then be calculated by Equation (56).

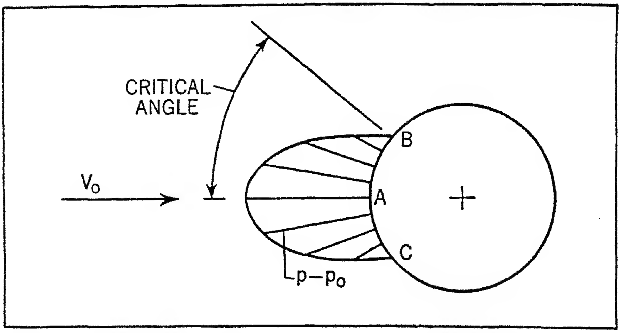


FIG. 63. Pressure distribution for two-dimensional flow around a cylinder.

Each opening in the tube can be connected to one side of a differential gage, as shown in Fig. 64*b*. The pitot tube in a stream of unknown direction can be rotated about its axis until the pressure at each hole is p_0, that is, until the differential pressure is zero. In this position the

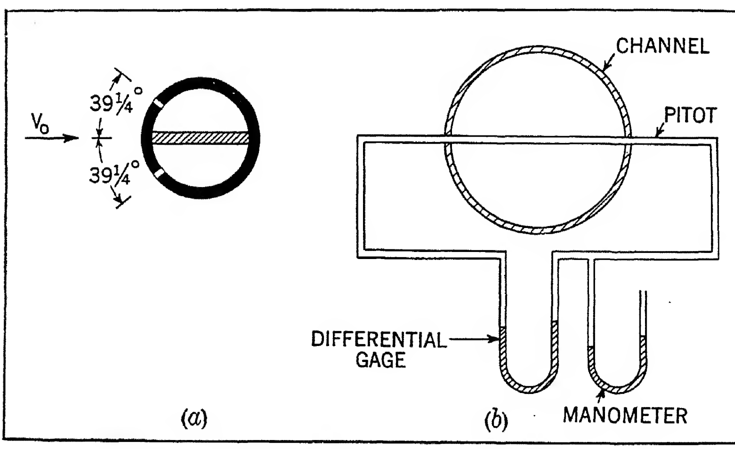


FIG. 64. Cylindrical or direction-finding pitot tube.

bisector of the angle between the holes gives the flow direction. Thus this pitot provides means for measuring the velocity vector in both magnitude and direction.

63. Venturi meter

If a direct weighing or a volumetric measurement of rate of discharge is not possible or convenient, then some secondary method might be used. A venturi (named after Venturi) meter, nozzle, orifice, or weir may be used for measuring or controlling the rate of flow.

The venturi meter, as indicated in Fig. 65, consists of a venturi tube and a suitable differential pressure gage. The venturi tube has a converging portion and a diverging portion. The function of the converging portion is to increase the velocity of the fluid and temporarily lower its static pressure. A pressure difference between inlet and throat is thus developed, which pressure difference is correlated with the rate of discharge. The diverging cone or diffuser serves to change the area of the stream back to the entrance area, and to reduce lost energy.

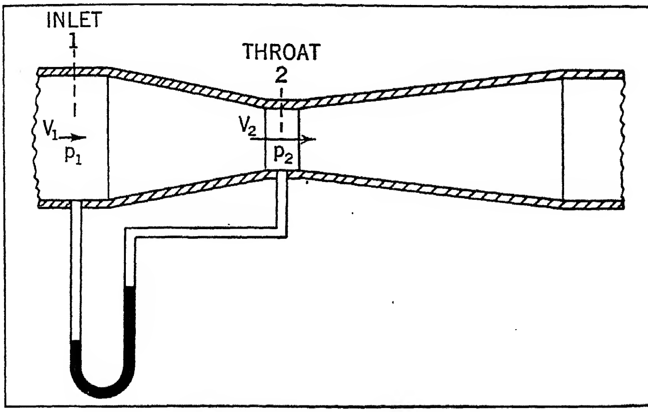


Fig. 65. Venturi meter.

Different venturi-tube proportions and different arrangements of pressure gages are found in practice. The diameter of the throat is usually between one-half and one-fourth of the inlet diameter. In many tubes the included angle of the entrance cone is about 21°, and the included angle of the exit or diffuser cone is between 5° and 7°. It is claimed that an included angle between 5° and 7° gives the lowest resistance at ordinary velocities. If the total loss across the venturi tube is not important, then the diverging cone is superfluous.

Using the notation shown in Fig. 65, assume incompressible flow and no friction losses. Then the energy equation becomes

$$\frac{p_1}{w} + \frac{V_1^2}{2g} = \frac{p_2}{w} + \frac{V_2^2}{2g}. \tag{57}$$

Use of the continuity equation $Q = A_1V_1 = A_2V_2$ with Equation (57) gives

$$\frac{p_1}{w} - \frac{p_2}{w} = \frac{V_2^2}{2g}\left[1 - \left(\frac{A_2}{A_1}\right)^2\right],$$

$$\text{ideal } Q = A_2V_2 = \frac{A_2}{\sqrt{1 - \left(\frac{A_2}{A_1}\right)^2}}\sqrt{2g\left(\frac{p_1 - p_2}{w}\right)} \tag{58}$$

Note that the specific weight w in Equations (57) and (58) is the specific weight of the fluid flowing through the meter. The venturi may be connected to a manometer containing a fluid which is different from that of the flowing stream. A measurement of the pressure difference

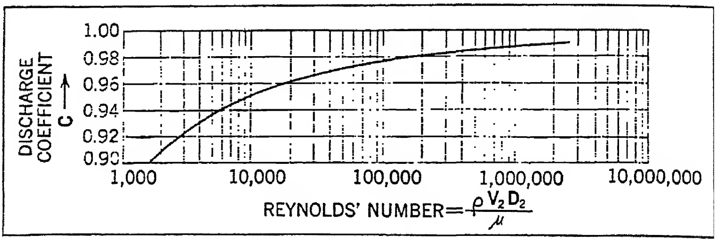


FIG. 66. Discharge coefficients C for venturi meters.[3] Ratio of inlet to throat diameter from 2 to 3.

$p_1 - p_2$ offers a method for determining the rate of discharge Q. Equation (58) is an ideal relation which assumes uniform velocity distributions at inlet and throat, and no friction. Differences between the actual and ideal flows can be taken into account by an experimentally determined discharge coefficient.

The actual discharge Q is commonly expressed as $Q = CA_2V_2$, where the discharge coefficient C is dimensionless. Dimensional analysis and dynamic similarity show that C is a function of Reynolds' number. Some experimental values of C are shown in Fig. 66. D_2 is the internal diameter at the venturi throat.

Figure 66 is only intended to give some idea of the order of magnitude of the discharge coefficient. For accurate test work it is advisable to calibrate the venturi meter in its actual service location. It is good technique to calibrate the meter by weighing the discharge during a measured time interval.

[3] Data adapted from *Fluid Meters, Their Theory and Application*, Part I, A.S.M.E., 1937, page 56.

64. Flow nozzles

Nozzles are employed to form jets and streams for a variety of purposes, as well as for fluid metering. Usually the term *flow nozzle* refers to a nozzle placed in or at the end of a pipe for purposes of metering. As illustrated in Fig. 67, the flow nozzle may be considered as a venturi tube that has been simplified and shortened by omitting the diffuser on the

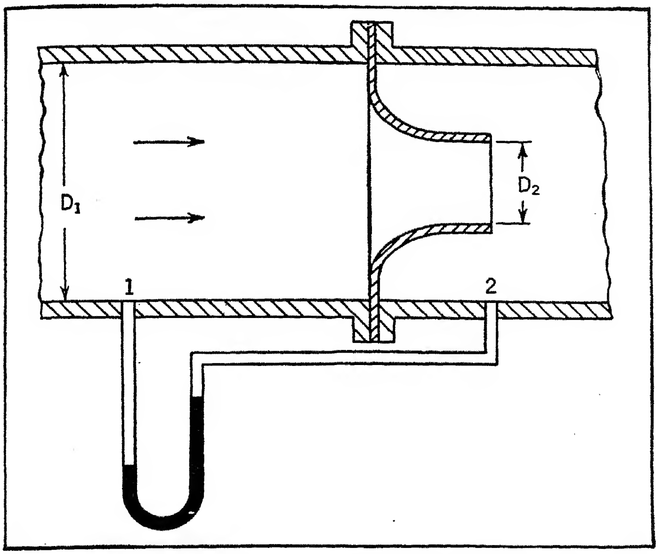


FIG. 67. Flow nozzle with differential gage.

outlet side. The final equation for the venturi meter can be applied to the flow nozzle, to give

$$\text{actual } Q = \frac{CA_2}{\sqrt{1 - \left(\frac{A_2}{A_1}\right)^2}} \sqrt{2g\left(\frac{p_1 - p_2}{w}\right)}. \tag{59}$$

The rate of discharge can be correlated with the measured pressure difference.

Several organizations have been sponsoring extensive programs to establish standards on flow nozzles and other fluid meters. These standards are helpful in forming the basis for acceptance tests on turbines, compressors, and other power equipment. New standards are published in the current literature as soon as they are sufficiently developed. One organization, the I.S.A. (International Federation of National Standardizing Associations), has standardized certain nozzle details and

correlated a great deal of experimental data.[4] For some time the A.S.M.E. (American Society of Mechanical Engineers) has been sponsoring a research program on nozzles which are designated as "long-radius low-ratio" and "long-radius high-ratio." One objective of the A.S.M.E. selection of nozzle shapes and installation details is to obtain discharge coefficients which are close to unity.

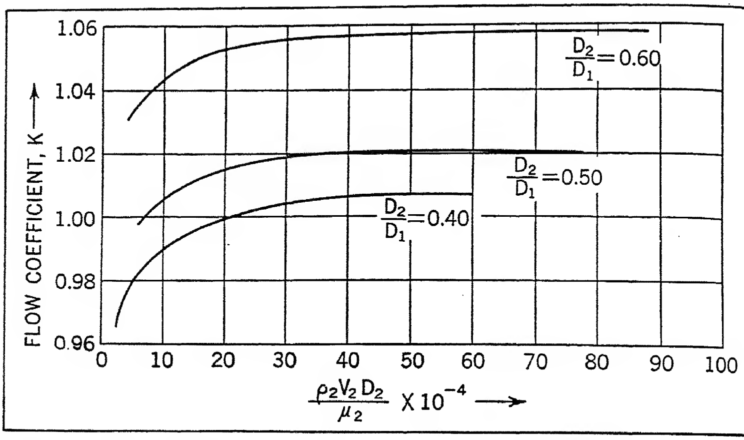


FIG. 68. Flow coefficients for A.S.M.E. long-radius high-ratio flow nozzles as a function of Reynolds' number.[5]

Some of the terms in Equation (59) are frequently grouped into a single term; this term is called the *flow coefficient K*, and is defined as

$$K = \frac{C}{\sqrt{1 - \left(\frac{A_2}{A_1}\right)^2}}. \tag{60}$$

The dimensionless term $\frac{1}{\sqrt{1 - \left(\frac{A_2}{A_1}\right)^2}}$ is commonly called the *velocity-of-approach factor*. Then Equation (59) can be written as

$$\text{actual } Q = KA_2 \sqrt{2g\left(\frac{p_1 - p_2}{w}\right)}. \tag{61}$$

[4] *Flow Measurement by Means of Standardized Nozzles and Orifice Plates*, in *Power Test Codes*, PTC 19.5.4-1940, A.S.M.E., 1940.

Standards for Discharge Measurement with Standardized Nozzles and Orifices, National Advisory Committee for Aeronautics Technical Memorandum No. 952, 1940.

[5] *Power Test Codes*, PTC 19.5.4-1940, A.S.M.E., 1940.

The numerical value of the dimensionless flow coefficient K is not necessarily an indication of accuracy. The coefficient K may be unity, less than unity, or greater than unity. Dimensional analysis and dynamic similarity show that K is a function of Reynolds' number and other factors. Some sample values are given in Fig. 68.

65. Jet flow

Some features of jet flow will be reviewed before discussing the practically important case of orifice meters. Figure 69 shows an orifice with a free discharge. Applying the energy equation between sections 1 and 2 for incompressible frictionless flow gives

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + h = \frac{p_2}{w} + \frac{V_2^2}{2g}. \tag{62}$$

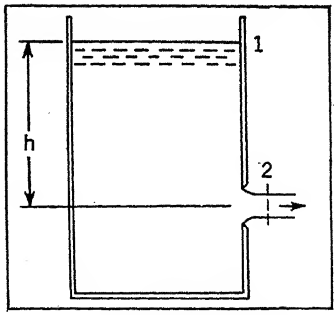


Fig. 69. Orifice with free discharge.

If the free surface 1 and the jet at section 2 each were open to the atmosphere, $p_1 = p_2$. If the area A_1 were large in comparison with A_2, the continuity equation shows that V_1 would be small in comparison with V_2. Assume that V_1 can be neglected. Then Equation (62) becomes

$$\text{ideal } V_2 = \sqrt{2gh}. \tag{63}$$

V_2 in Equation (63) is an "ideal" velocity because of the conditions taken. Equation (63) is identical with the relation for the velocity attained by a freely falling body.

Because of friction the actual jet velocity is less than the ideal. The ratio of the actual velocity to the ideal velocity is called the *coefficient of velocity* C_V. The actual $V_2 = C_V \sqrt{2gh}$. The streamlines converge as they approach the orifice, and continue to converge beyond the orifice. At a certain distance from the plane of the orifice, the jet has a minimum area where all the streamlines are parallel. This effect is shown in the actual photograph in Fig. 70. This minimum section of the jet is termed the *vena contracta*. The ratio of the minimum jet area to the orifice or opening area is called the *coefficient of contraction* C_c. If A is the orifice area, an ideal rate of discharge Q can be expressed as

$$\text{ideal } Q = A\sqrt{2gh}.$$

Friction and contraction being taken into account, the actual discharge is found to be

$$\text{actual } Q = C_V C_c A \sqrt{2gh} = CA\sqrt{2gh}, \tag{64}$$

where C is the discharge coefficient. For sharp-edged circular orifices C_V ranges from about 0.95 to about 0.99, and C_c ranges from about 0.61 to 0.72. The discharge coefficient C is probably the one of most practical use, and the one that can be found most readily experimentally.

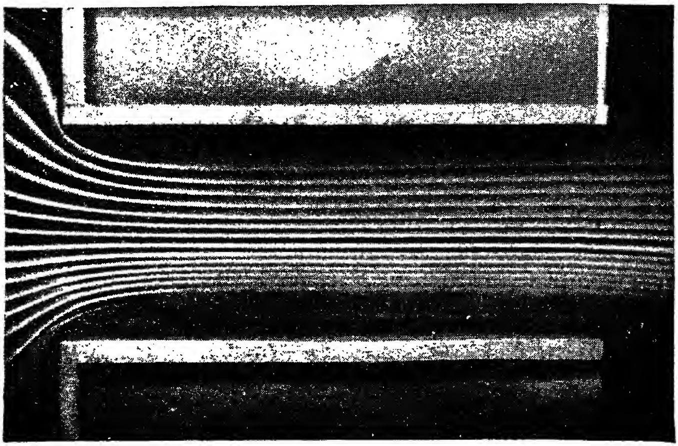


FIG. 70. Example of the converging of streamlines in a jet. Flow is from left to right.

66. Orifice meter

The orifice is possibly one of the oldest devices for measuring and controlling the flow of fluids. The thin-plate or sharp-edged orifice is

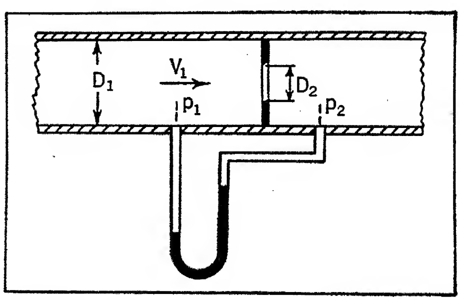
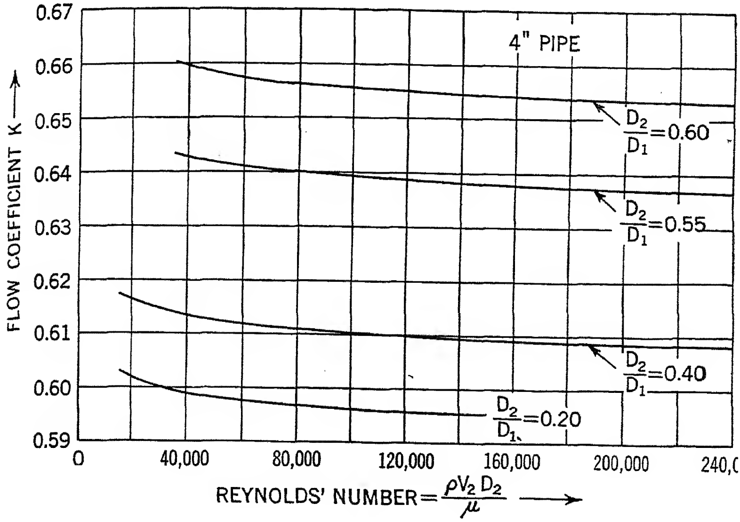


FIG. 71. Orifice meter with differential manometer.

frequently employed for metering. A common arrangement consists of a thin flat plate which is clamped between the flanges at a joint in a pipe line; the flat plate has a circular hole concentric with the pipe. Pressure

connections, or "taps" for attaching separate pressure gages or a differential pressure gage, are made at holes in the pipe walls on both sides of



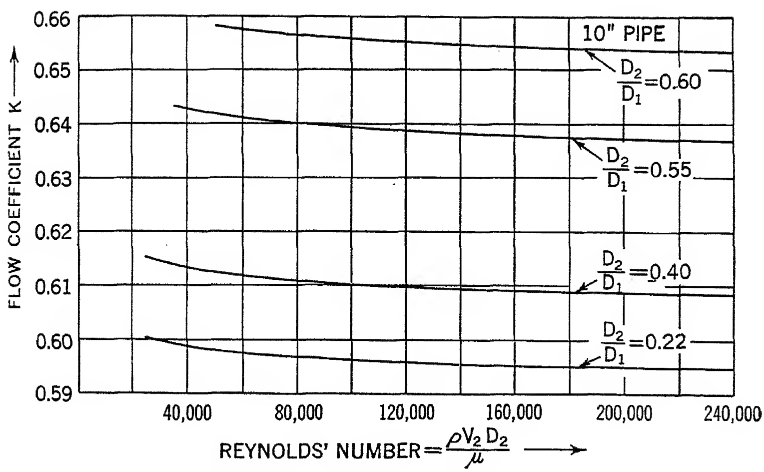


FIG. 72. Flow coefficient K for orifice meters with *vena contracta* pressure taps.[6]

the orifice plate. In some cases the pressure connections are integral with the orifice plate. The inlet edge of the orifice is sometimes rounded.

[6] Data from *Fluid Meters, Their Theory and Application*, A.S.M.E., 1937.

Because round-edged orifices are difficult to specify and reproduce, their use is usually restricted to experimental work or special instruments.

Using the notation shown in Fig. 71, the equation for the venturi tube can be applied to the orifice meter, to give

$$\text{actual } Q = \frac{A_2 C}{\sqrt{1 - \left(\frac{A_2}{A_1}\right)^2}} \sqrt{2g\left(\frac{p_1 - p_2}{w}\right)}. \tag{65}$$

where A_2 is the orifice area, A_1 the pipe area, and C the discharge coefficient. The term $\left(\frac{p_1 - p_2}{w}\right)$ is commonly replaced by a differential head h, and the term $\frac{C}{\sqrt{1 - \left(\frac{A_2}{A_1}\right)^2}}$ by the flow coefficient K, to yield

$$\text{actual } Q = KA_2\sqrt{2gh}. \tag{66}$$

Note that the specific weight w in Equation (65) is the specific weight of the fluid flowing through the orifice.

The A.S.M.E. Reports on Fluid Meters give experimentally determined values of C and K as functions of Reynolds' number for various orifices and orifice installations. Some sample values are shown in Fig. 72. In Fig. 72 V_2 is the average velocity in the orifice opening. The term *vena contracta taps* signifies that the center of the downstream pressure tap is placed at the minimum pressure position, which is assumed to be at the *vena contracta*. The center of the inlet pressure tap is located between $\frac{1}{2}$ and 2 pipe diameters from the upstream side of the orifice plate; usually a distance of one pipe diameter is employed. Figures 68 and 72 show a general tendency found in the behavior of various flow meters, a tendency for the flow coefficient to approach a constant value as the Reynolds' number is increased.

67. Direct determination of coefficients for flow meters[7]

It is common practice to express flow coefficients as a function of Reynolds' number. This practice is followed in an earlier chapter dealing with pipe friction factors, in the present chapter in dealing with meter coefficients, and will be followed in subsequent chapters in dealing with other flow coefficients. Reynolds' number provides a pertinent, convenient, and dimensionless parameter for correlating data for similar flow conditions with the same and different fluids.

[7] Some of the material in this article appeared in *Direct Determination of Coefficients for Flow Meters*, in *Instruments*, Instruments Publishing Co., Pittsburgh, June, 1942, page 196.

The discharge coefficient C or the flow coefficient K for a fluid meter can be determined if $N_R = \rho(VD/\mu)$ is known. In many practical cases, however, the problem is to determine the rate of discharge Q or the velocity V. At first thought it might appear that a trial-and-error method of computation is necessary; a possible procedure is to assume first a velocity (or N_R), next determine the corresponding flow coefficient, and then check, or repeat until the calculated velocity (or N_R) agrees with the assumed value. Such a trial-and-error method is not really necessary.

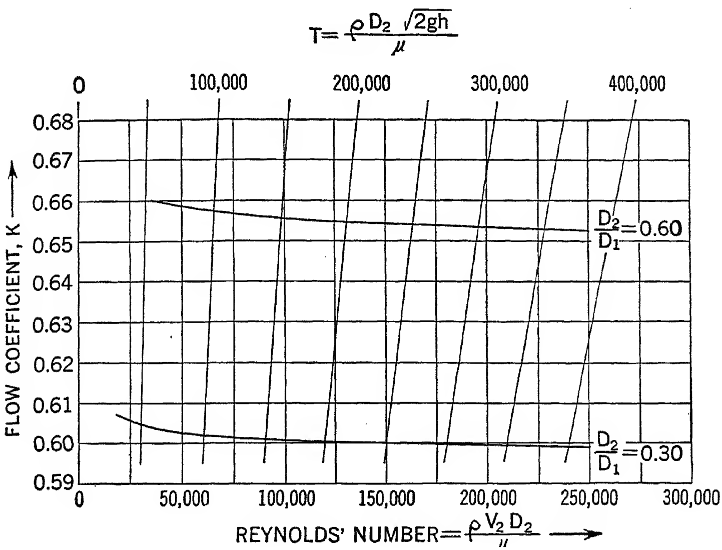


FIG. 73. Example of chart for direct determination of flow coefficients. Data are for orifice meters in pipe of 6-inch diameter, *vena contracta* taps.

The case of an orifice meter will be taken as an illustration. The flow coefficient K was plotted against $N_R = \rho(D_2V_2/\mu)$, where V_2 is the actual average velocity in the orifice opening. Equation (66) states that $Q = KA_2\sqrt{2gh}$. Thus, V_2 can be written as $V_2 = K\sqrt{2gh}$. Since K is a function of V_2, and V_2 is a function of h, K can be expressed as a function of h. Reynolds' number can be written as

$$N_R = \frac{\rho D_2 V_2}{\mu} = \frac{\rho D_2 K \sqrt{2gh}}{\mu}$$

or

$$\frac{N_R}{K} = \frac{\rho D_2 \sqrt{2gh}}{\mu} = T. \tag{67}$$

The dimensionless ratio N_R/K will be called a T number. It is to be noted that D_2, h, ρ, and μ in the T number are quantities which are directly measured or established before making a discharge calculation. If suitable T lines are available on a diagram of K against N_R, then K can be determined *directly* once T is established.

Figure 73 shows an example. The procedure is to calculate first the ratio $\rho D_2 \sqrt{2gh}/\mu$, follow along the T line to the curve applying, and then read K or N_R directly. No trial-and-error computation is necessary. T lines can be easily added to a diagram. The equation of a constant T line is N_R = constant K. This is the equation of a straight line, and only requires two points for plotting. The same general method can be applied to plots of K versus N_R for flow nozzles and venturi meters. The sound unifying features of a diagram of K against N_R can be supplemented with T lines to permit a direct determination of coefficients for flow meters.

68. Weir

Weirs are used for the measurement and control of the flow of water in open channels. Using the notation shown in Fig. 74, the height H of

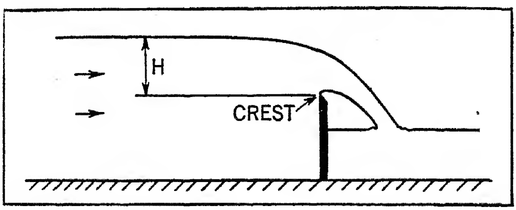


FIG. 74. Weir.

the undisturbed water level above the crest of the weir is correlated with the volumetric discharge per unit time. It is necessary to ventilate the weir, to allow air to pass freely under the jet. Since H in general is small, its measurement has to be carried out accurately. A common method is to have a micrometer screw with a sharp conical point, or hook, entirely submerged in the water. This point is screwed up until it touches the surface, and the height H observed.

Using the notation shown in Fig. 75, the ideal velocity of a stream filament or layer issuing from a weir notch at a distance z below the undisturbed surface is

$$\text{ideal } V = \sqrt{2gz}. \tag{68}$$

If this layer thickness is dz and its width L, the ideal volume rate of flow for an element is

$$\text{ideal } dQ = LVdz = L\sqrt{2gz}dz. \tag{69}$$

Integrating Equation (69) between the limits $z = 0$ and $z = H$ gives

$$\text{ideal } Q = L\sqrt{2g}\int_0^H z^{1/2}dz = \tfrac{2}{3}L\sqrt{2gH^3}. \tag{70}$$

The actual rate of flow is less than the ideal, so that

$$\text{actual } Q = \tfrac{2}{3}CL\sqrt{2gH^3}, \tag{71}$$

where C is a discharge coefficient. Applying a similar method of analysis to other types of weirs gives the general relations shown in Fig. 76.

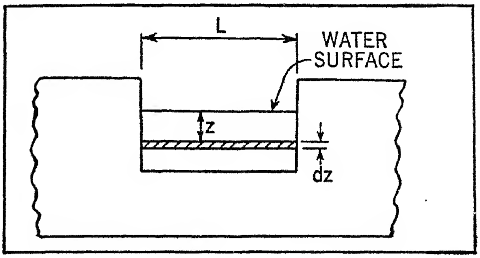


FIG. 75. Front view of flow over a rectangular weir.

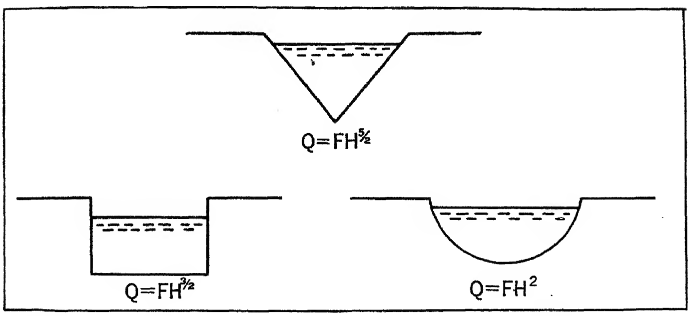


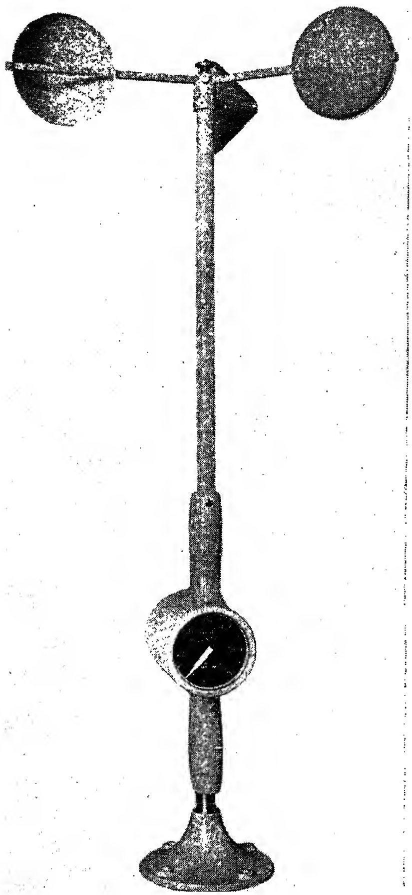
FIG. 76. General form of relations for different types of weirs. F is a proportionality factor.

The best plan is to calibrate the weir in place, under the same conditions for which it is to be used. If this procedure is not possible, weir coefficients might be estimated after consulting reports of original experimental investigations.

69. Anemometers

There are some mechanical devices which consist primarily of a rotating element whose angular speed of rotation is correlated with the linear velocity of flow. These devices require calibration. When used for air flow, such devices are commonly termed *anemometers*, whereas when used for water flow they are called *current meters*. The rotating

element may consist of a series of cups or a series of vanes (similar to a windmill), mounted on a shaft held in bearings. The rotating shaft may be connected to some sort of revolution counter, and an observation made



Courtesy of Julien P. Friez & Sons.

FIG. 77. Anemometer for measuring wind speed.

of the number of revolutions in a measured time interval. Figure 77 shows an anemometer used for measuring wind speed. Movement of the cups by the wind is transmitted to the horizontal cross arms to rotate the vertical shaft.

Various forms of hot-wire anemometers have been developed for the measurement of velocity and velocity fluctuations. Such instruments have been used primarily in air streams. In the hot-wire anemometer there is a wire (frequently platinum) of small diameter which is heated by passing an electric current through it. When exposed to moving air the heated wire cools and consequently its electrical resistance changes. The velocity of flow is correlated with certain electrical measurements. Two

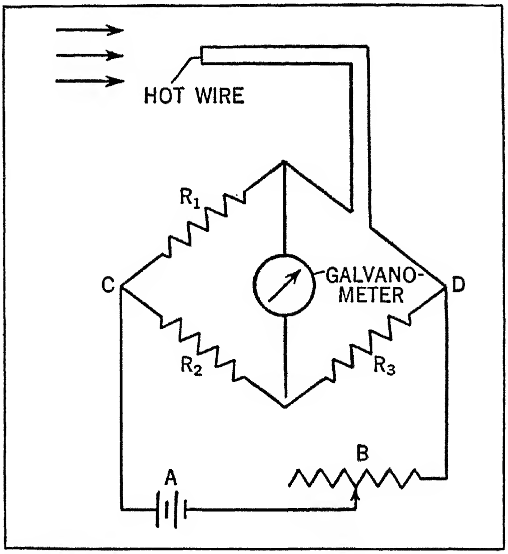


FIG. 78. Schematic arrangement of constant-voltage hot-wire anemometer.

types of circuits have been devised: (*a*) a constant-voltage circuit, and (*b*) a constant-resistance circuit.

The constant-voltage arrangement is illustrated diagrammatically in Fig. 78. R_1, R_2, and R_3 are electrical resistances in the Wheatstone bridge circuit. The hot wire forms the remaining resistance in the bridge. *A* represents a battery, and *B* a variable electrical resistance. The voltage across the bridge (across points *CD*) is kept constant after the circuit is so adjusted that the galvanometer indicates zero current when the heated wire is in stationary air. As the air flows, the hot wire cools, the resistance changes, and the galvanometer deflects. The galvanometer deflection is correlated with the air velocity by calibration. This method is particularly adapted for low air velocities.

Figure 79 shows a schematic arrangement of the constant-resistance hot-wire anemometer. As air flows, the temperature of the wire is kept

constant by varying the resistance in the battery across the bridge. The bridge voltage is varied so that the galvanometer reading remains zero. The voltmeter reading is correlated with the air velocity by calibration. Further details regarding hot-wire anemometers can be found in the

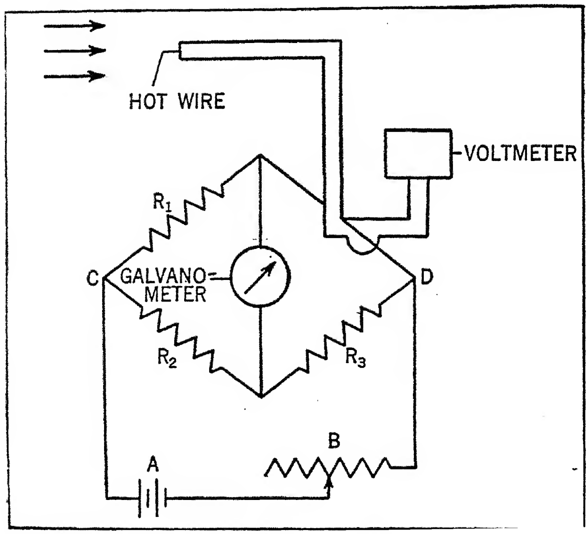


FIG. 79. Schematic arrangement of constant-resistance hot-wire anemometer.

publications of Dryden, Kuethe, Wattendorf, and others (see the references at the end of this chapter).

70. Some volumetric meters

There are certain flow meters which are sometimes designated as *volumetric meters*, or, briefly, *volumeters*. In this type of instrument the fluid passes through the meter in successive and more or less completely isolated volumes. The rotary meter illustrated in Fig. 80 is one example. The two lobes, impellers, or rotors are mounted on parallel shafts, and rotate in opposite directions. A pair of timing gears, located at one end of the shafts, maintains the proper relation between the lobes throughout rotation. Fluid enters the space between a lobe and the case, and moves from inlet to discharge. An indication of the volumetric displacement (such as cubic feet) can be obtained by means of a revolution counter connected to the shaft of one lobe.

Such a rotary unit can be employed as a meter by itself, or it can be

employed to calibrate such other flow meters as orifices and flow nozzles. In general, it is not difficult to calibrate a flow meter handling a liquid; an accurate calibration can be made by weighing the discharge during a time interval. The calibration of flow meters handling air or gas, however, presents a much more difficult problem. Rotary meters of the type shown in Fig. 80 can be adjusted for calibrations within an accuracy

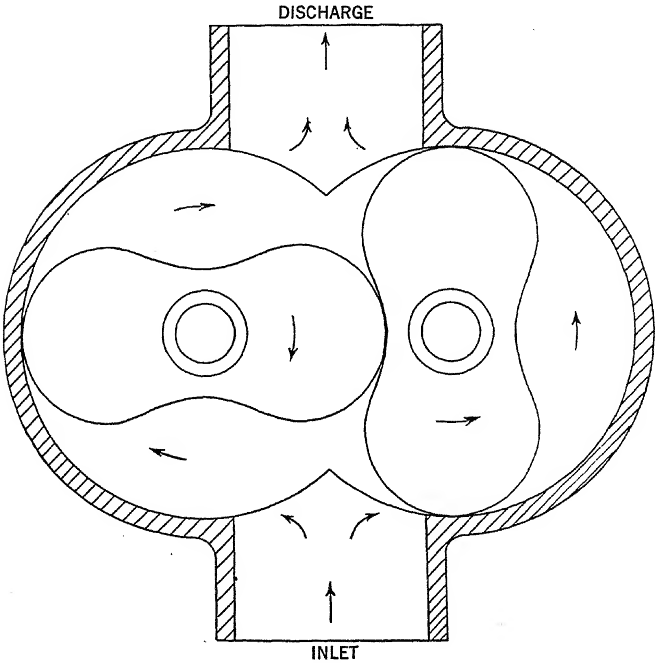


Fig. 80. Two-lobe rotary meter.

of 0.5 per cent. The meter to be calibrated is placed in series with the rotary meter.

The *nutating-disk meter* or *wobble-plate meter* is another example of a volumetric meter. Such a device is frequently used to meter the water supply for domestic use. The movable element in Fig. 81 is a circular disk attached to a central ball. A pole or shaft A is fastened to the ball; this shaft is held in an inclined position by a cam or roller. The disk is mounted in a chamber which has a spherical side wall, a conical top surface, and a conical bottom surface. The disk is prevented from rotating about its own axis by a radial slot which engages a radial partition in the chamber. Liquid enters through an opening in the spherical wall on one side of the partition, and leaves on the other side of the

partition. As liquid flows through the measuring chamber (alternately above and then below the disk), the disk wobbles or executes a nutating motion (nodding in a circular path without revolving about its own axis). The shaft A generates a cone with the apex down. The top of shaft A actuates a revolution counter (not shown in Fig. 81) through a crank and

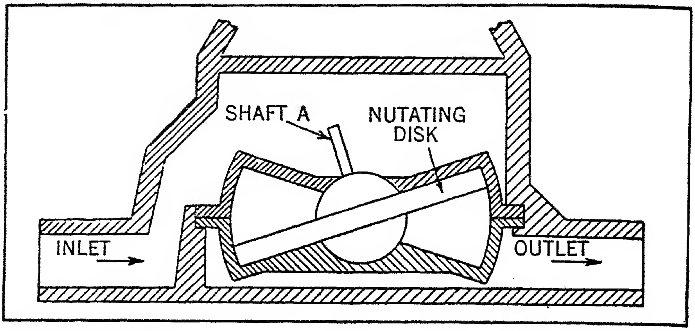


FIG. 81. Diagrammatic sketch of the measuring chamber in a nutating-disk meter.

a set of gears. The dial indication is correlated by calibration with the volume (as gallons or cubic feet) that has passed through the meter.

71. Salt-velocity method of measuring discharge

The salt-velocity method of measuring the rate of water flow in a conduit is based on the fact that salt in solution increases the electrical conductivity of water. Salt, or a salt solution, is introduced into the conduit at some convenient point. One or more pairs of electrodes are arranged at one or more sections, and the flow of the solution recorded by electrical instruments. The time required for the salt solution to travel a measured distance is determined. The volume rate of flow is calculated from these data and the dimensions of the conduit.

72. Pipe elbow as a flow meter

Experimental investigations reported by Lansford[8] led to the conclusion that the ordinary commercial pipe elbow (90° bend) may be used successfully as a flow meter. A diagrammatic sketch of this elbow meter is shown in Fig. 82. The dynamic action of the flowing fluid causes a difference between the pressures at the inside and outside curves of the elbow. Pressure connections from the inside and outside curves of the elbow are led to a differential pressure gage. The differential pressure is correlated with the average velocity of flow in the connecting pipe or the

[8] *The Use of an Elbow in a Pipe Line for Determining the Rate of Flow in the Pipe* by W. M. Lansford, University of Illinois Engineering Experiment Station Bulletin No. 289, December 22, 1936.

rate of discharge. For accurate results an elbow meter should be calibrated in the actual service location. Some of the favorable features of an elbow meter are low initial cost, small cost of upkeep, and *no* additional resistance to flow due to the elbow being converted into a meter.

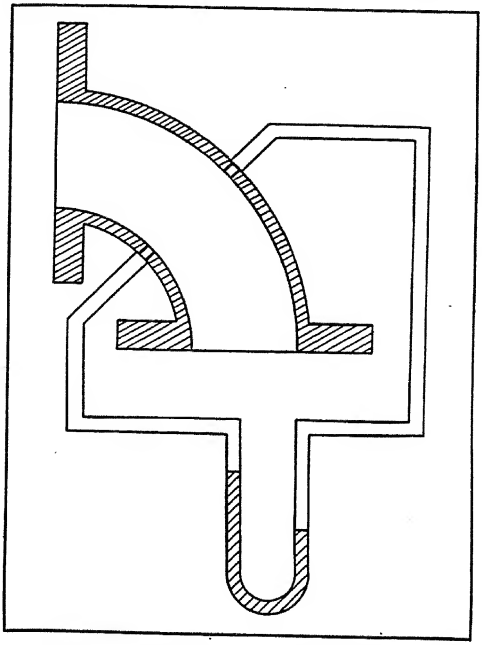


FIG. 82. Diagrammatic sketch of elbow meter.

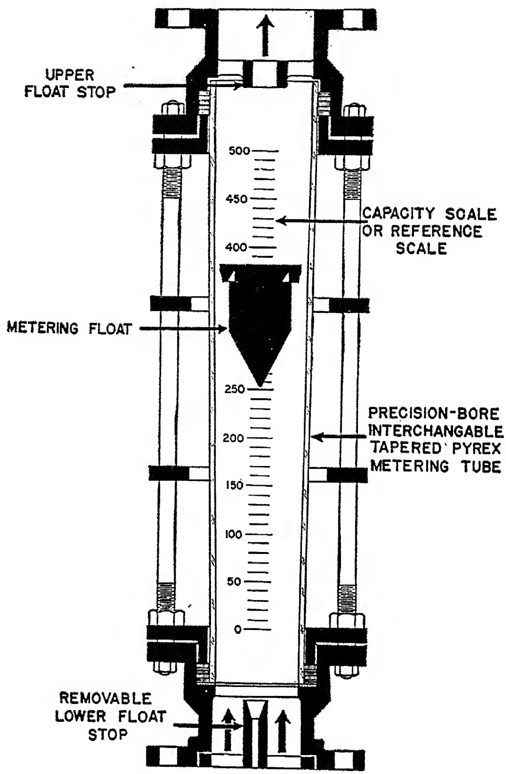
73. Rotameter

The *rotameter* is a flow meter which derives its name from the fact that a rotating free float is the indicating element. As shown in Fig. 83, a transparent tapered tube is set in a vertical position with the large end at the top. The fluid flows vertically upward through the tube. Inside the tapered tube is a freely suspended float of plumb-bob shape. When no fluid is passing, the float rests on a stop at the bottom end. As flow commences, the float rises toward the larger end of the tube, to make a passage for the fluid. The float rises only a short distance if the rate of flow is small; the float rise is greater for a higher rate of flow. There is a corresponding float position for each flow. Sometimes this type of instrument is classified as an *area type* flow meter, because of the varying annular space between the float and tube. Slantwise slots are cut in the

head of the float, to cause the float to rotate and to maintain a position in the center of the tube.

74. Capillary-tube viscometer

A viscometer or viscosimeter is an instrument for measuring viscosity. The operation of a viscometer depends upon the existence of laminar flow under certain controlled conditions. An ideal viscometer would be one in which the flow involved is completely determined by the fluid



Courtesy of Fisher & Porter Co.

FIG. 83. Rotameter.

viscosity. In actual viscometers this ideal is not completely realized, and it is necessary to apply certain correction factors or to calibrate the instruments with fluids of known viscosity.

Probably the best scientific method for determining dynamic viscosity is the *capillary-tube* or *transpiration* method. In this method a measure-

ment is made of the time required for a certain amount of fluid to flow through a capillary (or small-bore) tube of known length and diameter under a measured, constant pressure difference. The Hagen-Poiseuille law can be applied, if the flow is laminar, to calculate the viscosity.

$$\mu = \frac{\pi \Delta p D^4}{128 Q l}. \tag{72}$$

If the pressure is measured at the ends of a tube, certain corrections for changes in velocity distribution and inlet loss must be made. The corrections depend upon the particular apparatus used. Since viscosity depends upon temperature, it is necessary to control and specify the temperature in all viscosity measurements.

75. Other types of viscometers

Figure 84 shows the essential features of the Couette or rotational type of viscometer. The fluid to be tested is placed in the annular space between two concentric circular cylinders. One cylinder is rotated with respect to the other. Measurements of torque and velocity gradient can be correlated with viscosity, as by a calibration with liquids of known viscosity. Effects of the ends of the cylinder must be considered or reduced if the readings of this device are to be used for accurate determinations of viscosity. A particular instrument of this type may be useful for relative measurements, such as in comparing the action of different fluids.

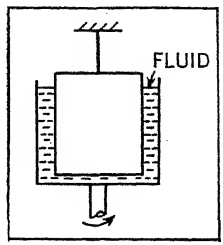


FIG. 84. Rotational type of viscometer. Fluid is placed in the annular space between two concentric circular cylinders.

Commercial variations of the rotational type of viscometer are the MacMichael and the Stormer. In the MacMichael instrument the outer cylinder is rotated at a constant speed, and the inner cylinder is supported by a torsion wire. A measurement of the angular twist of the wire is used to obtain a reading proportional to the viscous forces acting. In the Stormer instrument the outer cylinder is stationary, and a constant torque is applied to the inner rotating cylinder by means of a weight and pulley arrangement. A measurement is made of the time required for a definite number of revolutions of the inner cylinder. Sometimes a horizontal flat disk, a circular open-bottom cup, or a fork is immersed in the fluid instead of an inner circular cylinder. Rotational viscometers are sometimes used for checking the manufacture of suspensions, paints, and food products.

In the falling-body type of viscometer, a body (a sphere for example) is allowed to fall through a mass of fluid in a cylindrical tube. The time taken for the body to fall a certain distance (in which the velocity

is constant) is measured, and these measurements are correlated with viscosity. This type of instrument can be calibrated with fluids whose viscosities are known. Fluids with unknown viscosities can then be tested within the range of calibration.

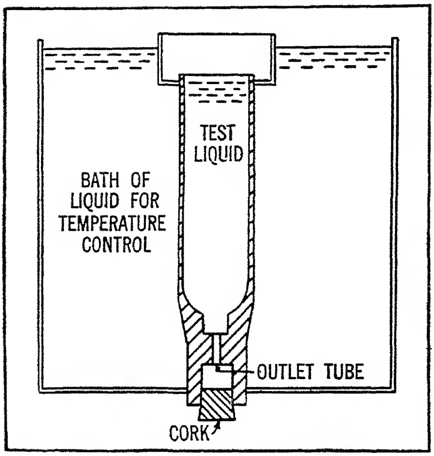


FIG. 85. Diagrammatic sketch of a Saybolt viscometer.

The Saybolt viscometer, as illustrated diagrammatically in Fig. 85, is widely used for comparative industrial purposes in this country, particularly for petroleum products and lubricants. Two instruments have been standardized by the American Society for Testing Materials;[9] these instruments are of the same general design, but of different dimensions. One is the Saybolt Universal, and the other is the Saybolt Furol (contraction for "fuel and motor oils").

The liquid to be tested is placed in the central cylinder, which has a short, small-bore tube and cork at its lower end. Surrounding the central cylinder is a bath of liquid

TABLE 7
CRANKCASE OIL CLASSIFICATION[10]

S.A.E. viscosity number	Range of Saybolt Universal readings, seconds			
	At 130° Fahrenheit		At 210° Fahrenheit	
	Minimum	Maximum	Minimum	Maximum
10	90	less than 120		
20	120	less than 185		
30	185	less than 255		
40	255			less than 80
50			80	less than 105
60			105	less than 125
70			125	less than 150

[9] *Standard Method of Test for Viscosity by Means of the Saybolt Viscometer* in *A.S.T.M. Standards*, 1939, Part III, page 216.

[10] Data from 1942 *SAE Handbook*. Society of Automotive Engineers, New York, page 485.

for maintaining the temperature of the test liquid. After thermal equilibrium is established, the cork is pulled, and the time required for 60 milliliters of the fluid to flow out is measured. This time, in seconds, is called the Saybolt reading. This Saybolt reading can be correlated

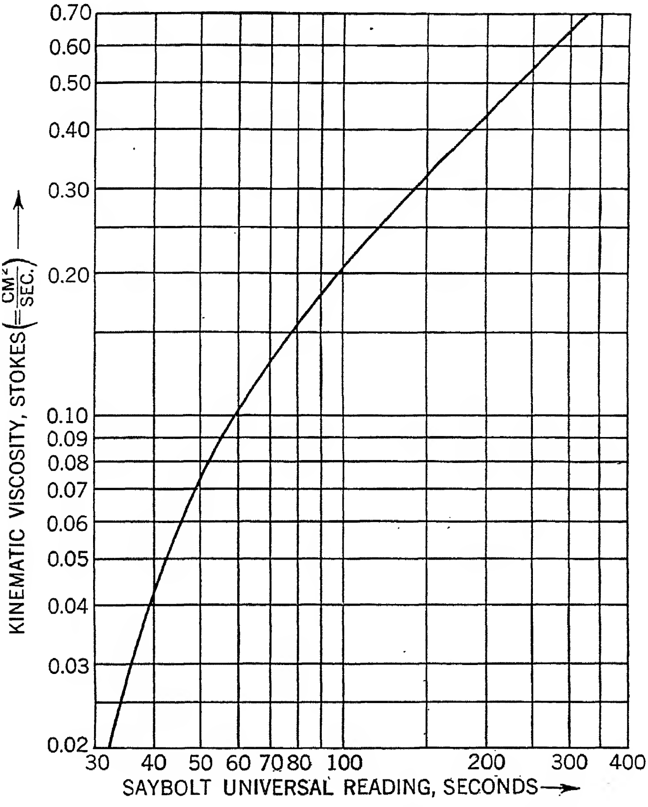


FIG. 86. Empirical correlation between Saybolt Universal Reading and kinematic viscosity at 100° Fahrenheit.

empirically with kinematic viscosity. One correlation at 100° Fahrenheit, based on experimental results,[11] is shown in Fig. 86.

The Society of Automotive Engineers has adopted certain S.A.E. numbers for some ranges of Saybolt readings. Table 7 shows S.A.E. numbers which are commonly used for designating crankcase oils, as far as viscous properties only are concerned.

[11] *A.S.T.M. Standards*, 1939, Part III, page 215.

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PROBLEMS

76. Air of standard density flows through a pipe. A differential manometer connected to a combined pitot tube shows a head difference of 0.180 inch of water. What is the velocity? Assume frictionless flow.

77. How can the discharge coefficient for a venturi meter be greater than unity even if there are frictional losses?

78. Benzene flows through a venturi meter having an inlet of 8-inch diameter and a throat of 3.5-inch diameter. The differential pressure is measured with a manometer having benzene on top of mercury. The mercury level in the throat leg is 4.0 inches above the mercury level in the inlet leg. If $C = 0.99$, what is the rate of discharge?

79. An A.S.M.E. long-radius flow nozzle, with a throat diameter of 2.75 inches, is installed in a water line of 6-inch diameter. A differential gage across the nozzle shows a pressure difference of 2.80 pounds per square inch. Calculate the rate of discharge if the discharge coefficient is 0.99.

80. The actual velocity in the *vena contracta* of a jet of liquid issuing from an orifice of 3-inch diameter is measured as 28.0 feet per second. If the head is 15 feet, and the actual discharge is 0.90 cubic feet per second, determine C_v and C_c.

81. Turpentine at 50° Fahrenheit (specific gravity = 0.87) flows through an orifice meter with *vena contracta* taps. The pipe diameter is 4.0 inches, and the orifice diameter is 1.6 inches. A differential gage across the meter indicates a pressure difference equivalent to a head of 8.0 inches of water. What is the rate of discharge?

82. A liquid (specific gravity = 0.96) flows through a tube of 0.25-inch diameter at a rate of 0.22 cubic inch per second. The measured pressure drop in a length of 22.0 inches is 0.031 pound per square inch. What is the dynamic viscosity of the liquid?

83. The Saybolt Universal reading for an oil at 100° Fahrenheit, having a specific gravity 0.92, is 106 seconds. What is the dynamic viscosity of this oil?

CHAPTER 9

Momentum Relations for Steady Flow

Every body continues in its state of rest or uniform motion in a straight line, except in so far as it doesn't.—A. S. EDDINGTON.[1]

Many important practical problems involve the flow *around* bodies as well as the flow *through* pipes and channels. Chapters 9, 10, and 11 deal with the dynamic forces acting between fluids and bodies. The present chapter includes examples like the jet, in which the fluid stream acting on a body has a finite, relatively small boundary, whereas Chapters 10 and 11 treat cases in which the body is completely submerged in a relatively large expanse of fluid.

Momentum relations are useful in attacking fluid flow problems because these relations give information about conditions along the boundaries. Certain conclusions about the flow can be drawn without a complete knowledge of the flow phenomena *inside* the boundaries.

76. Momentum change

The linear momentum of a body is defined as the product of its mass m and its linear velocity V. The change in linear momentum with respect to time t is equal to the resultant force F acting on the body, or

$$F = \frac{d}{dt}(mV). \tag{73}$$

If m is constant, then

$$= m\frac{dV}{dt} \qquad \text{or} \qquad Fdt = mdV, \tag{74}$$

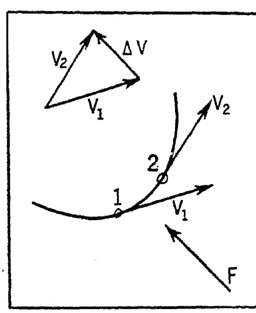


FIG. 87. Change in velocity of a particle.

where Fdt is called the *impulse* and mdV is called the momentum change.

In Fig. 87 a particle at point 1 has the velocity V_1. After a time interval Δt, the particle reaches point 2 with the velocity V_2. The force F causing the velocity change ΔV is in the direction of ΔV. Force, velocity, and momentum are vector quantities, that is, each is a quantity

[1] *The Nature of the Physical World.* Macmillan, New York, 1933.

having both a magnitude and a direction. Mass and time, however, are scalar quantities; each is completely specified by a magnitude only.

For the stream shown in Fig. 88, the infinitesimal volume at point B has a length ds and a cross-sectional area A. The mass of this infinitesimal volume is $w/g\, Ads$. The velocity of a fluid particle may vary with space and time. For steady flow, the velocity is a function of space or

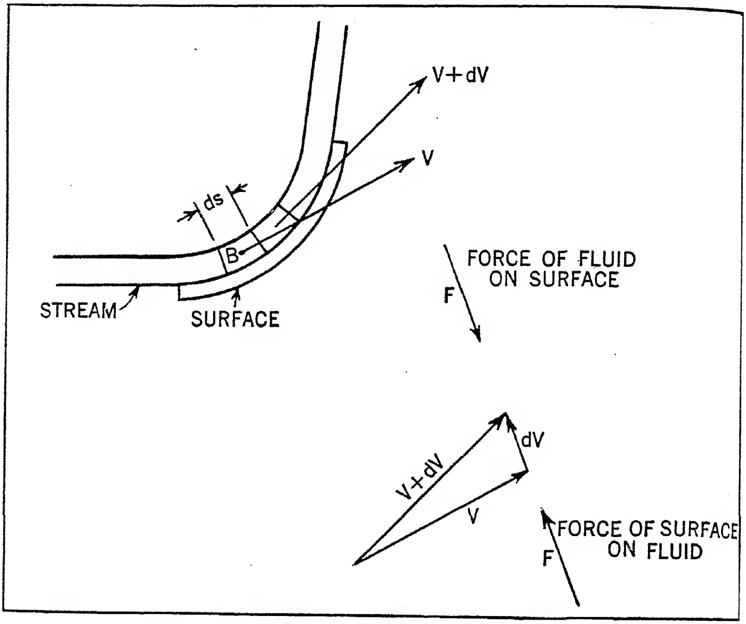


FIG. 88. Action between stream and curved surfaces.

distance alone (the velocity at any particular point does not change with time). For steady motion, the acceleration dV/dt may be written

$$\frac{dV}{dt} = \frac{dV}{ds} \cdot \frac{ds}{dt} = V\frac{dV}{ds}. \tag{75}$$

The force causing the velocity change dV is

$$F = m\frac{dV}{dt} = \frac{w}{g} Ads\, V\frac{dV}{ds},$$

$$F = \frac{w}{g} AVdV = \frac{w}{g} QdV = \rho QdV, \tag{76}$$

where Q is the volumetric rate of flow. The product ρQ is the mass of fluid flowing per unit time (such as slugs per second). In words, the

force equals the mass rate of flow times the velocity change. The fundamental relation in Equation (76) is a *vector* relation (F is a vector, and dV is a vector). In some cases it is convenient to deal with components. For two-dimensional flow in the xy plane, let F_x and F_y be the components of the resultant force F. Then

$$F_x = \rho Q(dV)_x \qquad F_y = \rho Q(dV)_y, \tag{77}$$

where $(dV)_x$ is the velocity change in the x direction, and $(dV)_y$ is the velocity change in the y direction. Since action is equal and opposite to reaction, the total force *of* the surface *on* the fluid is equal in magnitude but opposite in direction to the force *of* the fluid *on* the surface, as illustrated in Fig. 88.

The following articles will cover some applications in order to illustrate the physical meaning of the foregoing fundamental concepts. Further applications will be given in the remainder of this book.

77. Force of open jet on fixed surface

Figure 89 shows an open jet deflected from its initial course by a fixed curved surface. The case will be taken in which the jet is tangent

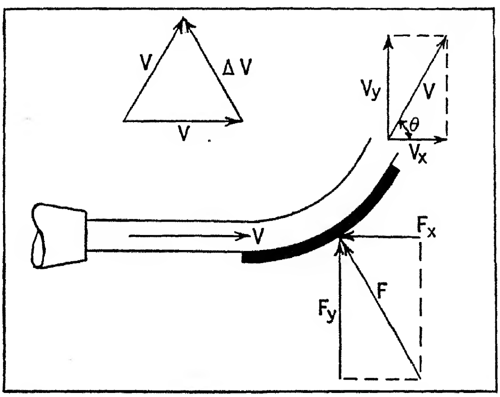


FIG. 89. Deflection of a jet by a fixed surface.

to a frictionless surface at the point of first contact' (no shock loss at entrance). The velocity at exit from the surface or vane is equal in magnitude to V, but the exit velocity has a different direction from that of the initial velocity. Taking the x-axis parallel to the jet, then the velocity change in the x-direction is $V - V\cos\theta$; the velocity change in the y-direction is $V\sin\theta$. The magnitudes of the force components are

$$F_x = Q\rho V(1 - \cos\theta), \qquad F_y = Q\rho V\sin 6. \tag{78}$$

The magnitude and direction of the resultant force F can be determined from the components. For example, the magnitude of F equals $\sqrt{F_x^2 + F_y^2}$

EXAMPLE. As shown in Fig. 90, a jet of water of 2-inch diameter strikes a fixed vane and is deflected 90° from its original direction. Determine the magnitude and direction of the resultant force on the vane or blade if the jet velocity is 50 feet per second.

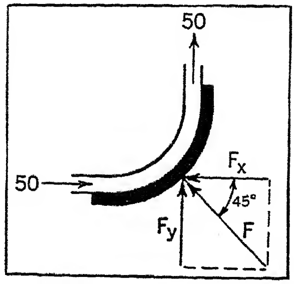


FIG. 90.

$$F_x = F_y = \frac{\pi}{4}\left(\frac{2}{12}\right)^2 50 \left(\frac{62.4}{32.2}\right) 50 = 105.7 \text{ pounds.}$$

$$\text{total force } F = \sqrt{F_x^2 + F_y^2} = \sqrt{2(105.7)^2} = 149.5, \text{ say } 150 \text{ pounds.}$$

The resultant force of 150 pounds acts at 45° with the original jet direction.

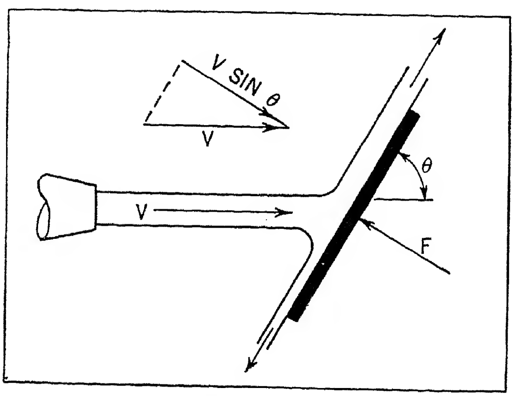


FIG. 91. Jet division by an oblique plate.

Figure 91 shows a jet striking a fixed plate which is not normal to the stream. The dimensions of the jet are small in somparison with those of the plate. The total change in linear momentum is normal to the

plate if friction is neglected. The total force F is normal to the plate, and has the magnitude

$$F = Q\rho V \sin\theta \tag{79}$$

The amount of fluid splitting in each direction can be determined by application of the continuity equation and the fact that there is no momentum change in a direction parallel to the plate.

78. Force of jet on a moving vane

A *single* moving surface or vane in the path of a jet deflects only that portion of the total discharge which overtakes the vane. If Q is the

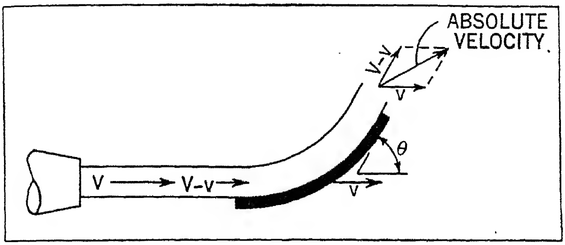


FIG. 92. Deflection of a jet by a moving vane.

discharge from the fixed nozzle in Fig. 92, the discharge overtaking the vane Q' is

$$Q' = Q\left(\frac{V - v}{V}\right), \tag{80}$$

where v is the absolute vane velocity. The fluid velocity relative to the blade at entrance is $V - v$. If there is no friction along the blade, the relative velocity of the fluid at exit has a *magnitude* equal to $V - v$. The absolute exit velocity of the fluid is the vector sum of $(V - v)$ and v. Taking the jet axis as the x-axis, the magnitudes of the force components for a *single* vane are

$$\begin{aligned} F_x &= \rho Q'[V - \{v + (V - v)\cos\theta\}], \\ F_y &= \rho Q'(V - v)\sin\theta. \end{aligned} \tag{81}$$

Since power is defined as the rate of doing work, the power $E = F_x v$. Note that the force F_x is in the direction of v. No work is done in a direction perpendicular to v.

79. Power developed by a series of vanes

A series of vanes or blades might be so arranged that the entire jet discharge is deflected by the vanes; if so, then Q' would equal Q. An arrangement of vanes on the periphery of a wheel, as in an ideal impulse

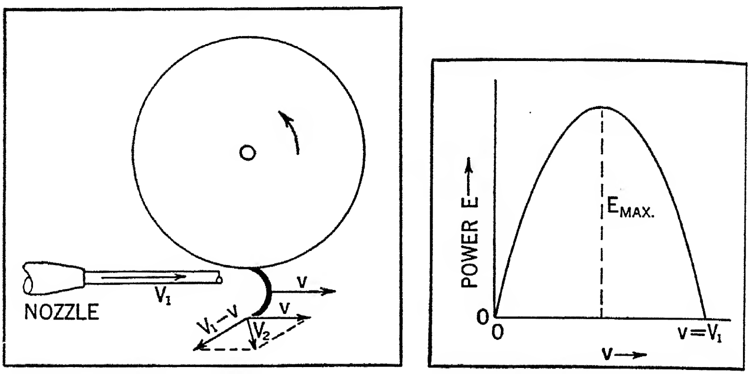
turbine, is one example. The useful torque producing force F_x for the arrangement in Fig. 93a is

$$F_x = Q\rho[V_1 - \{v + (V_1 - v)\cos\theta\}]. \tag{82}$$

The power E becomes

$$E = F_x v = Q\rho[V_1 v - \{v^2 + (V - v)v \cos\theta\}]. \tag{83}$$

For a given Q and jet size, the ideal power developed by the wheel, as expressed by Equation (83), will depend only on v. The power variation with v, as indicated in Fig. 93b, shows a maximum value of power for a



(a) (b)

FIG. 93. Deflection of a jet by a series of vanes, as in an ideal impulse turbine.

certain value of v. This maximum value $E_{\max.}$ can be found by differentiating E with respect to v, and setting the result equal to zero:

$$\frac{dE}{dv} = \frac{d}{dv}[Q\rho(V_1 - v)v(1 - \cos\theta)] = 0.$$

Solving for v gives the result that $v = V_1/2$ for maximum power.

80. Force on confined streams

When a stream of fluid is confined, forces caused by static pressure changes may exist as well as dynamic forces caused by velocity changes. A study of the flow in a reducing bend will illustrate the general method of force analysis. Figure 94 shows the forces acting *on* the body of fluid in the bend. The pressure forces p_1A_1 and p_2A_2 act on the fluid, in addition to the force R exerted by the pipe walls. The resultant of all these three forces equals the rate of change in linear momentum of the stream. If the inlet direction is taken as the x-direction, then the force components in the x- and y-directions are

$$p_1A_1 - R_x - p_2A_2 \cos\theta = Q\rho[V_2 \cos\theta - V_1],$$
$$R_y - p_2A_2 \sin\theta = Q\rho V_2 \sin\theta.$$
$$R_x = p_1A_1 - p_2A_2 \cos\theta - Q\rho(V_2 \cos\theta - V_1). \tag{84}$$
$$R_y = p_2A_2 \sin\theta + Q\rho V_2 \sin\theta. \tag{85}$$

The total force R of the bend on the fluid equals $\sqrt{R_x^2 + R_y^2}$.

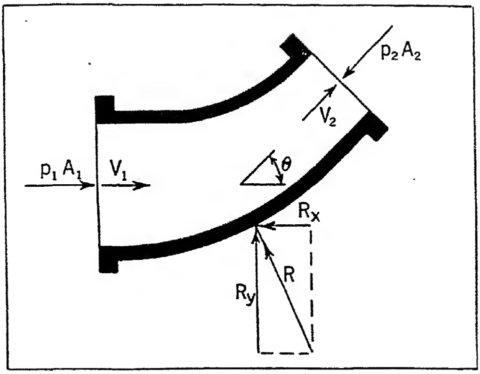


FIG. 94. Forces acting on the fluid in a reducing bend.

81. Torque developed in a rotating channel

The centrifugal pump and the reaction turbine are examples in which there is a flow in a channel as the channel rotates about an axis. The previous momentum relations can be extended for an analysis of such flows. Moments of momenta can be formed in the same manner that moments of forces are formed.

Consider a body of mass m moving at a radius r about an axis of rotation. Let V_t be the tangential component of the resultant velocity V of the body. The relation given by Equation (73) can be extended, by the multiplication of each side of the equation by r, to give

$$T = \frac{d}{dt}(mrV_t), \tag{86}$$

where T is the torque (tangential force times radial distance) or moment producing the momentum change. The product mrV_t is called *angular momentum* or *moment of momentum*. The radial component of linear momentum has no moment. In words, torque equals time rate of change in angular momentum. If m is constant,

$$T = m\frac{d}{dt}(rV_t). \tag{87}$$

Next, consider the steady flow through a rotating channel. Let the subscripts 1 and 2 refer to conditions at two different radii. Then V_{t_1} is the tangential velocity at radius r_1, and V_{t_2} is the tangential velocity at radius r_2. As in the foregoing articles, ρQ is the mass of the fluid flowing per unit time. Then Equation (87) becomes

$$T = Q\rho[V_{t_1}r_1 - V_{t_2}r_2]. \tag{88}$$

Equation (88) is sometimes called Euler's turbine relation.

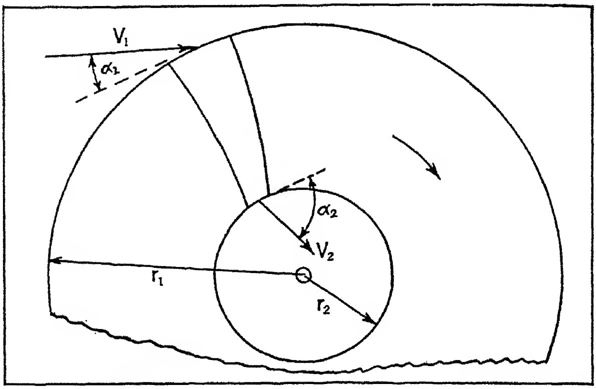


Fig. 95. Rotating channel, as in a turbine runner.

V_1 is the absolute entrance velocity to the channel on the rotating wheel in Fig. 95. V_2 is the absolute exit velocity from the channel. The torque acting on the wheel is then

$$T = Q\rho[V_1r_1 \cos \alpha_1 - \qquad \cos \alpha_2]. \tag{89}$$

Torque times angular velocity of the wheel gives power. In actual practice it is very difficult to predict and measure velocities in both magnitude and direction. Relations such as Equation (89) are useful in some respects, however, in indicating general features. For example, there is the question of the most favorable exit conditions. If α_2 is varied from 0° to 90°, Equation (89) shows that cos α_2 will be zero when α_2 is 90°; the torque and power reach a maximum for this variation when α_2 is 90°.

82. Borda mouthpiece

In Fig. 96 the tube extends from the wall of the container into the body of the fluid; such a re-entrant tube is commonly called a Borda mouthpiece. A discussion of the Borda mouthpiece is useful in showing how foregoing relations can be applied in attacking some problems. Let

A be the area of the opening, and A_c the area of the jet as it leaves the opening. The problem is to determine the value of the contraction coefficient $C_c = A_c/A$.

Two equations are available: an energy equation and a momentum relation. As shown in Article 65, Equation (63), if the fluid is regarded as incompressible and frictionless, the energy equation indicates that $V = \sqrt{2gh}$. Thus

$$Q = A_c \sqrt{2gh}. \tag{90}$$

At the level of the tube there is a pressure force equal to whA which acts towards the left on the container. An equal but opposite force acts

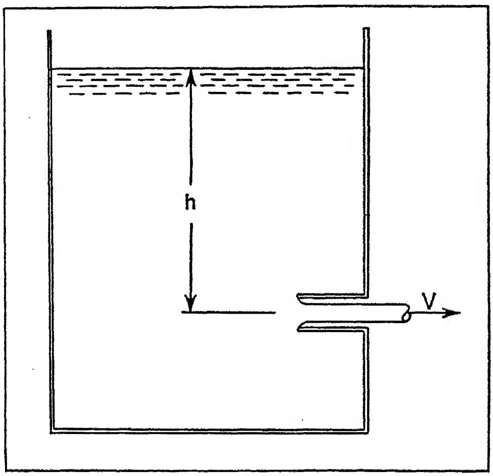


FIG. 96. Borda mouthpiece.

towards the right on the fluid, to increase the velocity from zero to V. Since force equals the time rate of change of linear momentum,

$$whA = w\frac{QV}{g}. \tag{91}$$

Combining Equations (90) and (91) shows that

$$\frac{A_c}{A} = 0.5 = C_c.$$

The contraction coefficient is 0.5.

83. Forced vortex and free vortex

Certain features of curved flow were brought out in foregoing articles. Since such flow occurs so frequently in practice, it is well to coordinate

the preceding discussions, and to investigate the important characteristics of two different types of circulatory flow. Attention will be restricted to steady, two-dimensional flow.

The energy equation applies to flow along a streamline. Equation (54), which gives the fundamental relation $dp = \rho dr \frac{V^2}{r}$, provides means for studying conditions in a direction normal to the streamlines. If the streamline is straight, the pressure change normal to the streamline is zero, because r is infinitely large. For streamlines of finite curvature, the pressure varies from p to $p + dp$ as the radius varies from r to $r + dr$. Since dp is positive if dr is positive, Equation (54) shows that the pressure increases for successive points from the concave to the convex side of the stream. The exact variation in pressure depends upon the variation in V with radius.

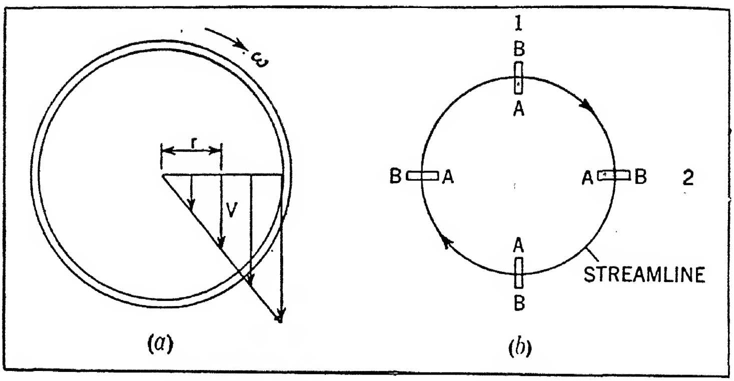


Fig. 97. Forced vortex, rotational motion.

Imagine a vertical cylindrical vessel with liquid rotating about a central vertical axis. The streamlines are concentric circles. If there is no motion of the fluid relative to the container, the fluid rotates like a solid body. Such a flow is called a *forced vortex;* a torque is applied to the body of fluid. Let ω (Greek letter omega) be the constant angular speed (radians per unit time) of the vessel, and r the radial distance to any point. As indicated in Fig. 97a, the linear velocity at any point in the fluid is given by the relation $V = r\omega$. The parabolic pressure distribution for this type of flow is given in Article 22, Chapter 2. The relations presented in Article 22 can be derived by starting with Equation (54). A forced vortex might be found in a rotating container fitted with radial vanes or partitions, in the rotating impeller of a centrifugal pump, or in the rotating runner of a turbine.

Sometimes the flow in the forced vortex is designated as *rotational.* The flow is *rotational* if each infinitesimal particle in the field of flow rotates about *its own axis.* As illustrated in Fig. 97b, the infinitesimal particle of fluid AB has a certain orientation at position 1. At position 2 the fluid particle has a different orientation; it has rotated 90° about its own axis.

Next, imagine a frictionless fluid moving in a horizontal circular path with *no* torque applied; such a flow is called a *free vortex.* The streamlines are concentric circles, but the velocity variation with radius is different

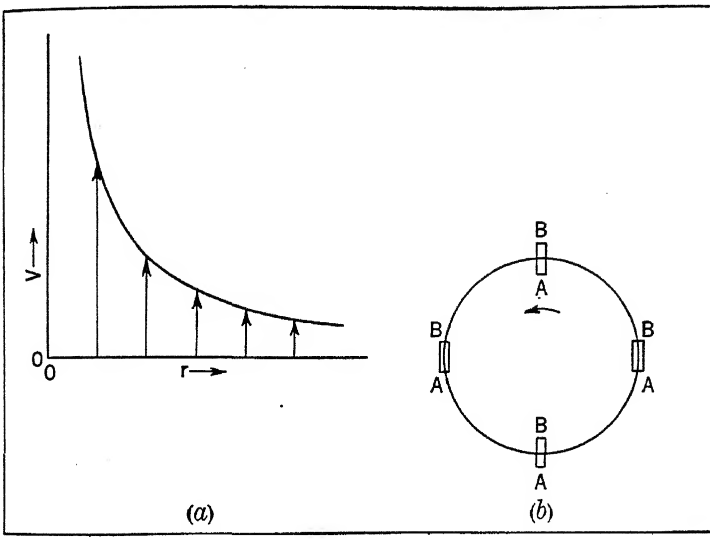


FIG. 98. Free vortex, irrotational motion.

from that of the forced vortex. If the torque T is zero, Equation (87) becomes (with $V_t = V$)

$$\frac{d}{dt}(rV) = 0.$$

Integration gives the relation $rV = K$, or

$$V = \frac{K}{r}, \tag{92}$$

where K is some constant. K can be determined from the known velocity and radius at a particular point. As indicated in Fig. 98a, the velocity decreases as the radius increases. The flow in a free vortex is *irrotational;* as shown in Fig. 98*b*, an *infinitesimal* fluid particle does not rotate about its own axis.

The pressure distribution in a free vortex can be determined by substituting the relation $V = K/r$ in Equation (54) and integrating between limits.

$$\int_1^2 dp = \rho \int_1^2 \frac{V^2}{r}\, dr,$$

$$p_2 - p_1 = \rho K^2 \int_1^2 \frac{dr}{r^3} = \left[\frac{1}{r_1^2} - \frac{1}{r_2^2}\right] \frac{\rho K^2}{2}. \tag{93}$$

Noting that $V_1 r_1 = V_2 r_2 = K$, Equation (93) can be written as

$$p_1 + \tfrac{1}{2}\rho V_1^2 = p_2 + \tfrac{1}{2}\rho V_2^2,$$

$$\frac{p_1}{w} + \frac{V_1^2}{2g} = \frac{p_2}{w} + \frac{V_2^2}{2g} = \text{constant}. \tag{94}$$

Equation (94) shows that the total energy in each stream-tube is the same as that in each of the other stream-tubes. No energy is added to the vortex by a torque, and no energy is disspated by friction.

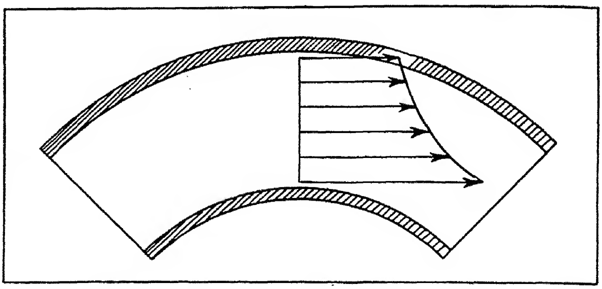


FIG. 99. Velocity distribution for two-dimensional flow in the main body of a stream flowing around a bend.

A free vortex might be found in the casing (volute) outside the rotating impeller of a centrifugal pump, or in the two-dimensional flow in the main body of a stream flowing around a bend, as shown in Fig. 99. Experimental results obtained from pitot traverses in pump volutes and in bends show fair agreement with the foregoing relations.

Equation (92) indicates that the velocity becomes infinitely large as the radius is decreased to zero. Infinite velocities are not reached in real fluids. In actual flows, viscous effects, which were neglected in deriving Equation (92), cause a portion of the fluid in the region near $r = 0$ to rotate like a solid body. Outside of this central core is a transition region; outside of the transition region is a free vortex. The velocity distribution and the pressure distribution for this combination flow are indicated in Fig. 100. The tornado and the waterspout (a tornado that forms at sea) are examples of such a circular whirl in the atmosphere.

There are indications that the horizontal velocities in some tornadoes have exceeded 200 miles per hour. The low pressure in the central core or "eye" and the high velocities of a tornado may be very destructive.

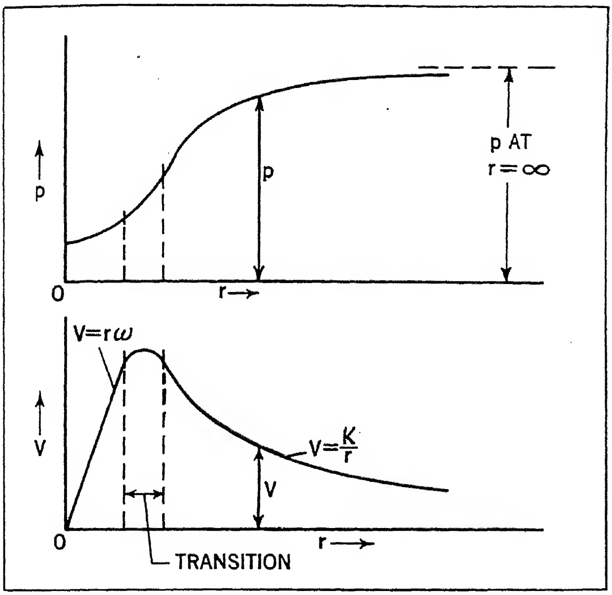


FIG. 100. Pressure and velocity distribution in a vortex flow.

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PROBLEMS

84. A jet of oil (specific gravity = 0.80) moving at 60 feet per second strikes a fixed flat plate normal to the stream. The dimensions of the jet are small in comparison with those of the plate. The rate of discharge is 0.20 cubic foot per second. What is the force on the plate?

85. A jet of lye solution (specific gravity = 1.10) 2 inches in diameter has an absolute velocity of 40 feet per second. It strikes a single flat plate which is moving away from the nozzle with an absolute velocity of 15 feet per second. The plate makes an angle of 60° with the horizontal. Calculate the total force acting on the plate, and the change in kinetic energy per unit time of the fluid.

86. A jet of glycerine $1\frac{1}{2}$ inches in diameter is deflected through an angle of 80° by a single vane. The jet velocity is 35 feet per second, and the vane moves

at 10 feet per second away from the nozzle in the direction of the entering jet. Determine the total force acting on the vane.

87. At the end of a water pipe of 4-inch diameter is a nozzle which discharges a jet having a diameter of $1\frac{1}{2}$ inches into the open atmosphere. The pressure in the pipe is 60 pounds per square inch gage, and the rate of discharge is 0.88 cubic foot per second. What is the magnitude and direction of the force necessary to hold the nozzle stationary?

88. A 90° bend is in a water pipe of 10-inch diameter in which the pressure is 100 pounds per square inch. The rate of flow is 40 gallons per second. What is the magnitude and direction of the force necessary to "anchor" the bend?

89. Refer to Fig. 91, page 118. If the angle θ is 60 degrees, what portion of the fluid leaving the nozzle will be deflected up, and what portion will be deflected down?

90. Derive Equation (17) by starting with Equation (54).

CHAPTER 10

Resistance of Immersed Bodies

The present chapter and the following discuss the forces acting on a body completely immersed in a relatively large expanse of fluid. Let V be the uniform, undisturbed velocity some distance ahead of the body at rest in Fig. 101. The fluid exerts a resultant force on the body; it is common practice to resolve this resultant force into two components. One component, along the line of V, is called the resistance or *drag*. The other component, at right angles to V, is called the *lift*. Drag and lift

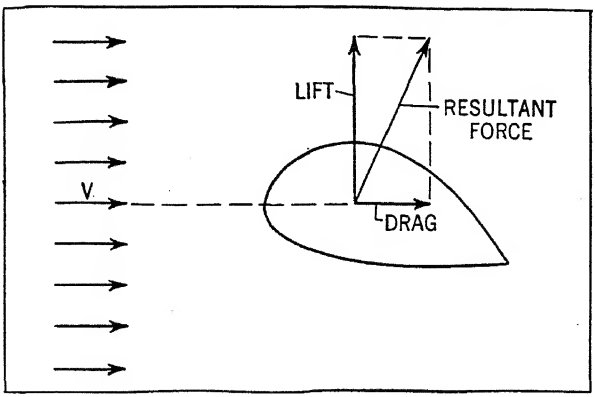


FIG. 101. Lift and drag forces.

studies are important in connection with various industrial processes, fluid machines (such as fans, pumps, and turbines), aeronautics, marine engineering, and other fields. The present chapter discusses drag; the next chapter will discuss lift. Both chapters will be limited to the steady flow of an incompressible fluid.

The force exerted by a fluid on a body depends only on the relative velocity between body and fluid, and not on the absolute velocity of either fluid or body. Figure 101 indicates one way of obtaining a certain relative motion. The same relative motion could be realized if the body were moving with the constant velocity V through a mass of fluid at rest some distance away from the body.

84. Deformation resistance at very low Reynolds' numbers

The flow, and consequently the drag, for a completely immersed body is determined by the viscous and inertia forces acting. It was pointed out in Article 49 that Reynolds' number $\rho Vl/\mu$ is proportional to the ratio of inertia forces and viscous forces. l is some characteristic length or dimension of the body.

Reynolds' number can become very small if the fluid is very viscous, if the velocity V is very low, or if the body dimensions are very small. The motion at a very small Reynolds' number, in which viscous forces predominate, is sometimes called a *creeping* motion. The corresponding resistance is sometimes called a *deformation* resistance. The resistance is due primarily to the deformation of the fluid particles; this deformation extends to large distances from the body.

85. Fluid resistance at high Reynolds' numbers

Reynolds' number, on the other hand, is high in many flows of practical importance. A high Reynolds' number, however, should not imply

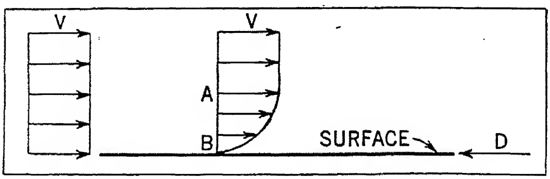


Fig. 102. Flow along a flat plate.

that the effect of viscosity on drag can be neglected. A fluid may have a very low viscosity; this small viscosity can have an appreciable effect, directly and indirectly, on the flow. The total resistance of a body at high Reynolds' numbers may be divided into separate components, one the *skin-friction* drag, and the other the *pressure* drag.

Figure 102 indicates a very thin, smooth plate parallel to the approaching flow; the velocity distribution is uniform ahead of the leading edge of the plate. If the fluid were nonviscous, the fluid would simply slip over the surface with the velocity V; at all points along the surface the velocity distribution would be uniform and identical with that ahead of the leading edge. No drag would result if the fluid were frictionless. A drag results, however, with real, viscous fluids. A very thin film of fluid adheres to the surface, whereas some distance normal to the plate the fluid has the velocity V. There is a velocity gradient from zero to V in the so-called *boundary layer* (as AB in Fig. 102). The boundary layer exists because the fluid is viscous; no boundary layer would exist if the fluid were nonviscous. Tangential or shear stresses are developed in the boundary layer because adjacent layers of fluid move

with different velocities. A force D is necessary to hold the plate in the stream, to overcome the so-called *skin-friction* drag. Some features of boundary layers were brought out in the chapter on pipe flow.

Pressure drag is the contribution due to pressure differences over the surface of the body. Viscosity effects can change the flow pattern, as to cause eddies, and thus influence the pressure distribution. As an example, the flow closes in smoothly behind a circular cylinder at very low velocities. If the flow were symmetrical as shown in Fig. 103, and if the fluid were frictionless, then the drag would be zero. For real fluids, the flow in Fig. 104a is obtained if the velocity is increased, or if the Reynolds' number is increased beyond 1. The fluid "separates" from the body, and an eddying wake is formed behind the body. Figures 104a,

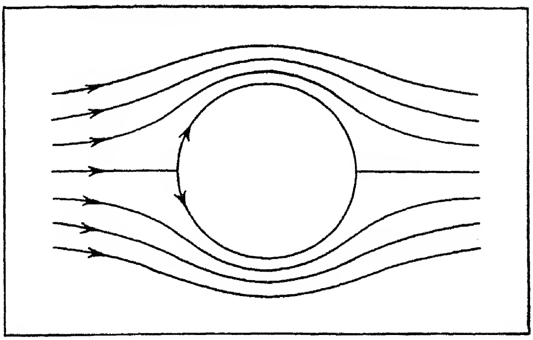
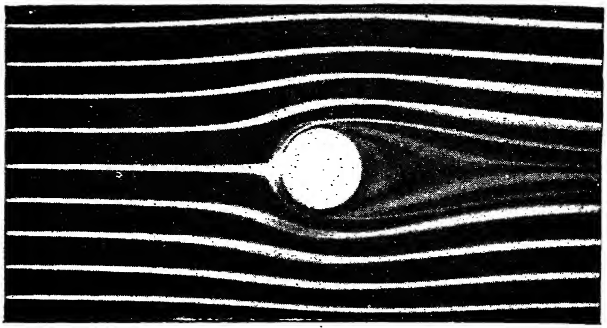


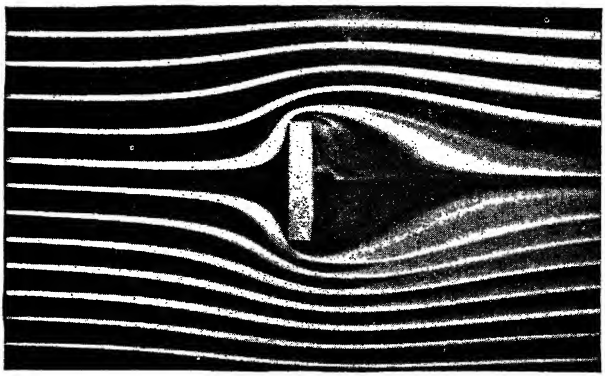
FIG. 103. Two-dimensional symmetrical flow around a circular cylinder.

104b, and 104c are photographs of the flow of air, a fluid which has a small dynamic viscosity. Figure 104a shows a difference between the streamlines upstream and downstream from the body. This difference indicates a variation in the pressure distribution around the body. The resultant of the pressure forces, over the body, in the line of undisturbed motion is the *pressure* drag force. The total drag of the body equals the sum of the skin friction drag and the pressure drag. The point of separation, and, therefore, the drag, may be different for different Reynolds' numbers.

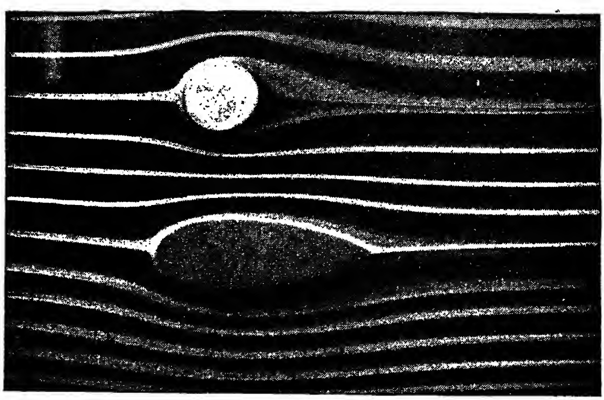
Figure 104b, for the flow around a flat plate, shows an appreciable wake behind the body; the pressure drag is relatively large. Figure 104c shows the flow around a cylinder and around a streamlined strut in the same stream. The streamlined section has practically no wake; the pressure drag is very small (as compared, for example, with that of a flat plate). The examples in Fig. 104 show that the phenomena giving rise to resistance are markedly affected by the *rear* of the body, as well as by the front of the body.



(a)



(b)



(c)

Fig. 104. Two-dimensional flow around various objects. Flow is from left to right.

86. Drag coefficients

It is customary to express the total drag D as

$$D = C_D \rho \frac{V^2}{2} \text{ (area).} \tag{95}$$

where C_D is the drag coefficient, a dimensionless ratio. The product $\rho V^2/2$ is the dynamic pressure. In some cases the area is taken as the projected area normal to the stream (as for spheres and cylinders). In other cases the area is taken as the largest projected area of the object; this area may be parallel or nearly parallel to V (as for airfoils and flat plates tilted slightly with respect to V).

For a completely immersed body, dynamic similarity and dimensional analysis show that C_D is a function of Reynolds' number $N_R = \rho Vl/\mu$. The following articles will give experimental values of C_D against N_R for various objects. For a particular problem, C_D can be determined once N_R is established, and the drag then computed by Equation (95). Drag coefficients are correlated in a manner similar to that used in correlating pipe friction factors.

87. Drag of sphere, disk, and bodies of revolution

Figure 105 shows a plot of C_D against Reynolds' number for spheres and circular disks. For the sphere, the point of separation, and hence the drag coefficient, is different for different Reynolds' numbers.

For N_R below about 0.4, experiments show that the drag coefficient for a sphere is $C_D = 24/N_R$. This relation, for laminar or viscous flow only, gives a drag value

$$D = C_D \frac{\rho V^2}{2} \text{ (area)} = \frac{24}{\rho V d/\mu} \frac{\rho V^2}{2} \left(\frac{\pi d^2}{4}\right)$$
$$D = 3\mu V \pi d \tag{96}$$

Equation (96) is commonly known as Stokes' law. Stokes' law is sometimes applied to problems dealing with the rise or fall of solid particles, drops, or bubbles through a fluid. The transportation of flue dust in the atmosphere, the transportation of silt in streams, the action in some ore-dressing processes, the measurement of dynamic viscosity by the "falling-ball" method, and the operation of air-lift pumps are examples. A solid particle in a fluid will rise or fall depending on whether the particle is lighter or heavier than the fluid. For steady vertical flow, the magnitude of the fluid resistance equals the difference between particle weight and the buoyant force. The limit of application of Stokes' law should be carefully noted.

Figure 106 shows values of C_D against N_R for two ellipsoids of revolution. The upper curve applies to a body in which the ratio of major

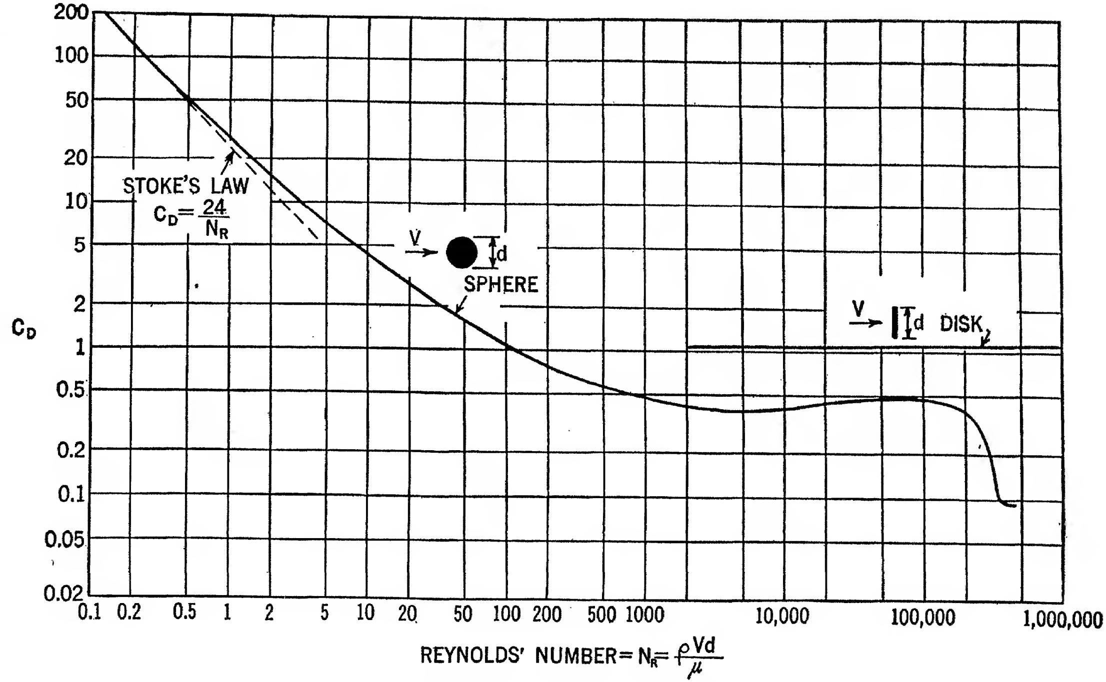


FIG. 105. Drag coefficients for sphere and circular disk. Area in the drag relation is projected area normal to the stream. (Data adapted from "Das Widerstandsproblem," by F. Eisner, *Proc. Third. Int. Cong. App. Mech.*, Stockholm, 1931.)

axis to minor axis is $\frac{4}{3}$; the lower curve applies to a body in which this ratio is 1.8. In Fig. 106, C_D is based on the area of the largest cross section normal to the undisturbed flow.

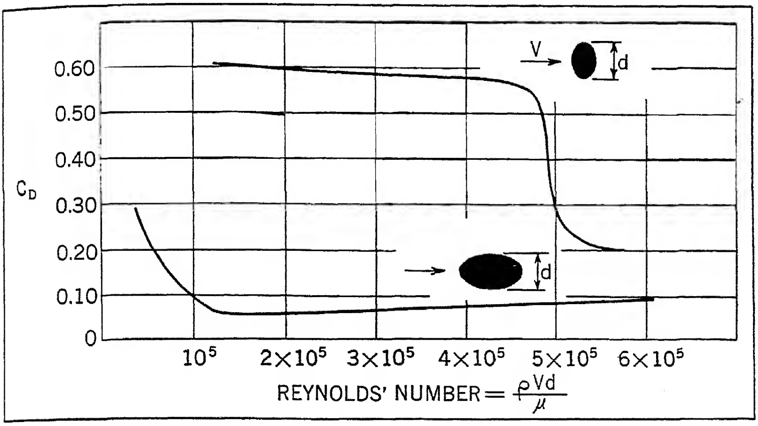


Fig. 106. Drag coefficients for ellipsoids of revolution.[1]

88. Drag of cylinder, flat plate, and streamlined sections

Figure 107 gives values of C_D against N_R for the two-dimensional flow around a cylinder and a flat plate. Two-dimensional flow can be obtained with an infinite length, or approached with a finite length with a flat plate at each end to prevent flow around each end. The drag coefficient for the flat plate is practically constant and nearly 2 for the range of N_R shown. In this range the fluid breaks away from the body, as shown in Fig. 104b.

Figure 108 shows values of C_D for plates of finite length; the fluid breaks away from the body as shown in Fig. 104b. The ratio x/y is sometimes called the *aspect ratio.* As the ratio x/y increases, the drag coefficient approaches the value given in Fig. 107.

Sometimes the statement is made that the resistance of a body varies as the square of the speed, and that the power required to overcome resistance varies as the cube of the speed. Such a statement is accurate only if the drag coefficient is constant in the range of speeds under consideration. Inspection of Figures 105, 106, and 107 shows that in some ranges the drag coefficient is practically constant, particularly at high Reynolds' numbers. In other ranges, however, the drag may vary approximately with the first power of the velocity. Stokes' law is an example.

[1] Data adapted from *Ergebnisse der aerodynamischen Versuchanstalt zu Göttingen* by L. Prandtl, vol. II, R. Oldenbourg, 1923.

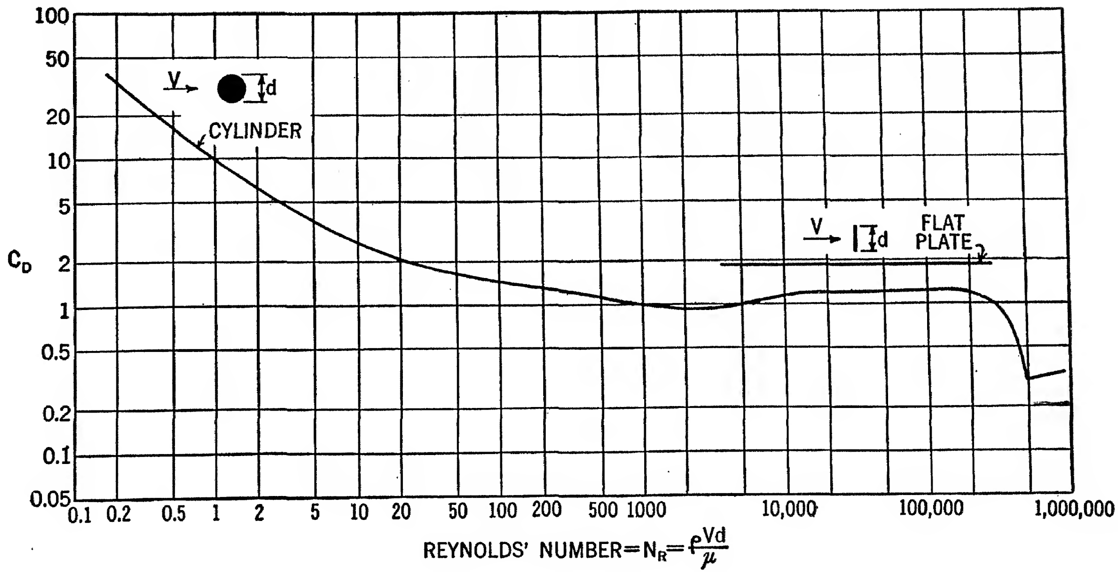


FIG. 107. Drag coefficients for two-dimensional flow around a cylinder and a flat plate. Area in the drag relation is projected area normal to the stream. (Data adapted from *Das Widerstandproblem* by F. Eisner, *Proc. Third Int. Cong. App. Mach.*, Stockholm, 1931.)

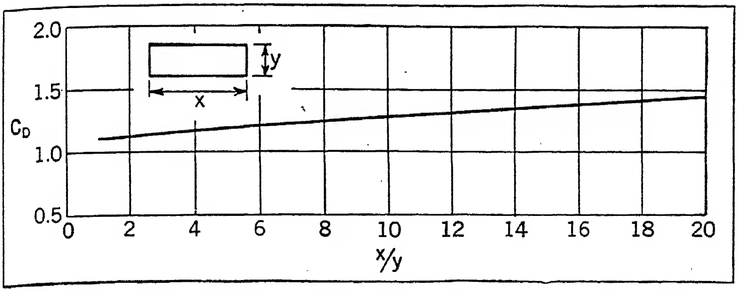


FIG. 108. Drag coefficients for a flat plate of finite length normal to flow.

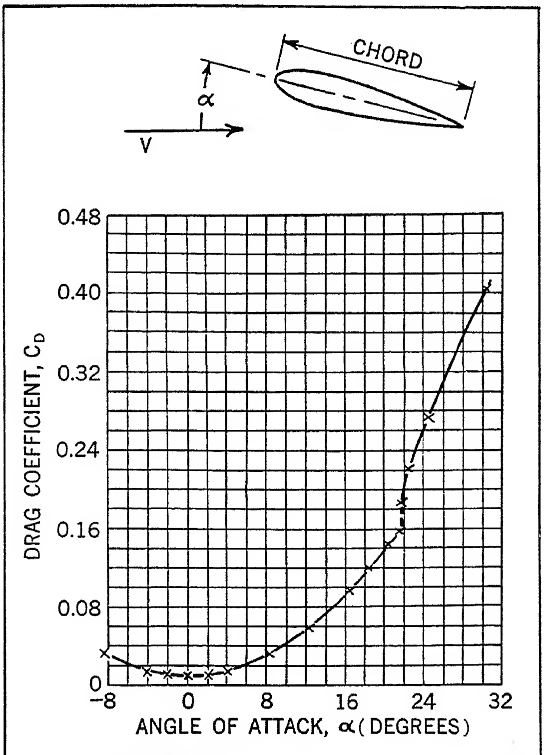


FIG. 109. Drag coefficient plotted against angle of attack for N.A.C.A. airfoil 0018. Reynolds' number = 2,970,000. (N.A.C.A. Technical Report No. 669, 1939, page 523.)

Streamlined sections of the general shape shown in the lower portion of Fig. 104c are desirable when low drag is of importance, as for airfoil, fan, and propeller sections. The gradually tapering tail serves to minimize the pressure or eddying drag. There are applications, however, as in heat-exchange equipment, in which it may be desirable to induce eddying flow in order to promote heat transfer. A large amount of data has been published on the drag and lift of streamlined sections. These data can be found in extensive reports, such as those of the N.A.C.A. (National Advisory Committee for Aeronautics). The variation of drag coefficient with angle of attack for one section is shown in Fig. 109. The area in the drag relation is the projected chord area. Note the low values of drag coefficient at low angles of attack.

89. Drag of smooth flat plate—skin friction

The boundary-layer thickness in Fig. 110 is zero at the leading edge, and increases with length along the plate. For some distance the

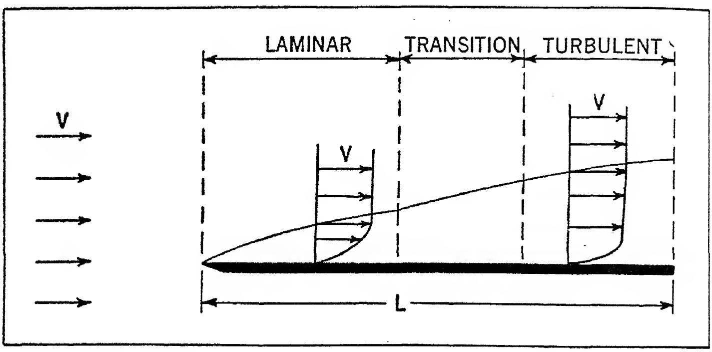


FIG. 110.—Boundary layers along a flat plate.

boundary-layer flow is laminar, with viscous forces predominating. The boundary-layer flow changes gradually from laminar to turbulent in a transition region. In the turbulent boundary layer, beyond the transition, the velocity profile is more blunt than the laminar profile. Within the turbulent boundary layer there is a very thin layer next to the plate in which the flow is laminar—this region is a laminar sublayer. The general problem of flow over a flat plate, or skin friction, is closely related to the general problem of pipe flow.

Figure 111 gives values of the skin-friction drag coefficient C_f against Reynolds' numbers for smooth flat plates. The total skin friction drag D is expressed as

$$D = C_f \frac{\rho V^2}{2} S, \tag{97}$$

where S is the wetted area. The L in Reynolds' number refers to the total length of the plate in the direction of flow. If the nature of the boundary layer is known, C_f can be found after N_R is determined, and the drag calculated by Equation (97). The dotted curves in Fig. 111 are for different transition conditions.

It is difficult to formulate perfectly general rules as to the nature of the boundary layer for all cases. The transition from laminar to turbulent flow is not sharp. The boundary layer depends upon the initial turbulence or fluctuations in the stream ahead of the plate, the shape of the leading edge, and the roughness of the plate. Turbulent boundary

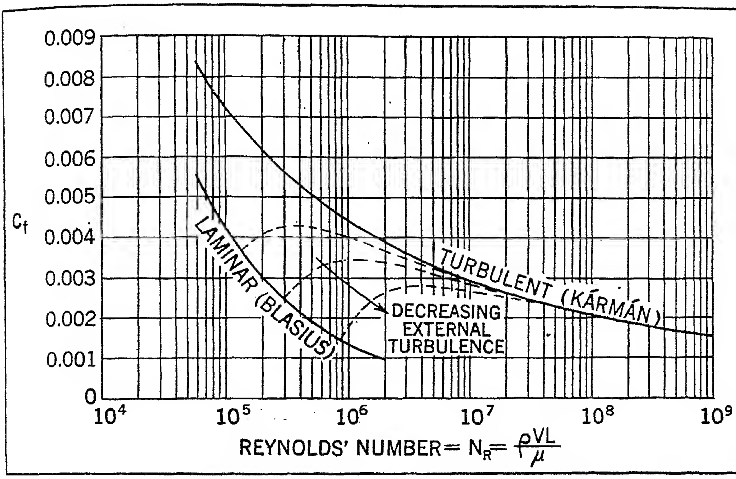


FIG. 111. Skin friction for smooth flat plates.[2]

layers at low Reynolds' numbers have been obtained with relatively blunt noses. Information about the particular application involved may be helpful in deciding whether the boundary layer is completely laminar or turbulent. For example, in calculations of skin friction for marine vessel hulls, it is common practice to regard the boundary layer as completely turbulent for usual conditions of operation.

90. Resistance of ships

The total resistance of a partially immersed or floating body, like a marine vessel, involves wave resistance in addition to the skin friction and the drag due to eddying flow. It was pointed out in Article 50 that both Froude's law and Reynolds' law of similarity are involved. Con-

[2] *Turbulence and Skin Friction* by T. von Kármán. *Journal of the Aeronautical Sciences*, vol. 1, No. 1, January, 1934.

sider a ship model, smaller in size than the prototype, to be tested in water. Reynolds' law indicates a model speed higher than that of the prototype, whereas Froude's law indicates a model speed lower than that of the prototype. Both laws might be satisfied by the use of different liquids, but to do this is not practical.

The difficulty is avoided by the following procedure. The model test is made on the basis of Froude's law, and the total drag measured. A calculated skin-friction drag is subtracted from the total model drag, to leave a *residuary resistance.* This residuary resistance is extended to the full-size ship. A calculated skin-friction drag is then added to the residuary resistance to give a value of total drag for the prototype ship. Further details can be found in references on naval architecture.

91. Kármán vortex trail

Certain phenomena associated with the flow around circular cylinders, elliptical cylinders, and flat plates are explained by reference to the

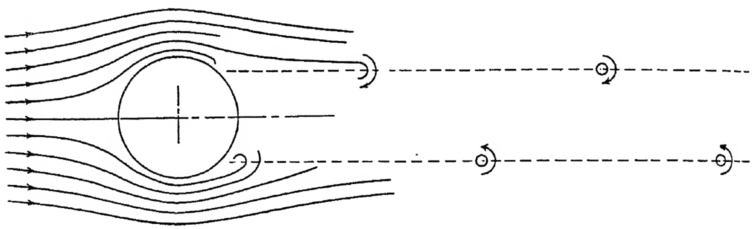


FIG. 112. Kármán vortex trail.

so-called *Kármán vortex trail.* Consider the flow around a circular cylinder. At Reynolds' numbers above about 20, eddies break off alternately on either side, in a periodic fashion, as indicated in Fig. 112. Behind the cylinder is a staggered, stable arrangement or trail of vortices. The alternate shedding produces a periodic force acting on the cylinder, normal to the undisturbed flow. The force acts first in one direction, and then in the opposite direction. Let f represent the frequency of this vibration in cycles per unit time, D the diameter, and V the undisturbed velocity. Experiments have shown values of the dimensionless ratio fD/V between 0.18 and 0.27.

If the frequency of the vortex peeling approaches or equals the natural frequency of the elastic system consisting of the cylinder and its supports, the cylinder may have a small alternating displacement normal to the stream flow. The vibration of some smokestacks, the vibration of some transmission lines, and the fatigue failure or progressive fracture of some transmission lines have been attributed to this resonance phenomenon. These vibrations might be avoided or reduced by changing the structural system or by the use of vibration dampers.

An aeolian harp, a box having strings which produce musical tones when placed in the wind, is an example of a device depending on the alternate peeling of vortices behind a circular cylinder. Other examples are the "singing" of telephone wires and the sound produced by wind as it whistles through long grass.

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PROBLEMS

91. A truck having a projected area of 68 square feet, traveling at 50 miles per hour, has a total resistance of 410 pounds. Of this amount, 25 per cent is due to rolling friction, and the remainder is due to wind resistance. What is the drag coefficient?

92. A spherical piece of quartz (specific gravity = 2.65) falls through a body of water. If the diameter is 0.0030 inch, what is the velocity of settling for steady viscous flow?

93. A steel ball (specific gravity = 7.85) 0.060 inch in diameter falls 0.16 feet per second through a mass of oil (specific gravity = 0.91). What is the dynamic viscosity of the oil? First assume steady viscous flow, and then check.

94. A spherical drop of water 0.002 inch in diameter exists in standard air. Assume steady laminar motion. Which way would the water drop move if it were in a vertically rising air current having a speed of: (a) 1.0 foot per second, (b) 0.6 foot per second, and (c) 0.2 foot per second?

95. A wire $\frac{3}{8}$ inch in diameter is exposed to a stream of carbon dioxide at a velocity of 100 feet per second. Undisturbed gas pressure is 14.7 pounds per square inch absolute, temperature is 90° Fahrenheit. What is the resistance per foot length of wire?

96. What is the total wind force on a rectangular sign board 1 foot by 4 feet, if the velocity is 20 miles per hour, the air is standard, and the flow is normal to the sign?

97. Find the resistance of a flat plate, 3 feet square, moving normal to itself at 20 feet per second, at 59° Fahrenheit and atmospheric pressure, (a) through air and (b) through water.

98. What horsepower is required to move a vertical automobile windshield, 4 feet wide and 2 feet high, at a speed of 60 miles per hour? What power is required if the speed is reduced to 30 miles per hour?

99. What is the wind force on a building, 120 feet high and 60 feet wide, if the air is standard and the velocity is 60 miles per hour? Assume that the drag coefficient is the same as that for a rectangular plate.

100. A streamlined train 400 feet long travels at 85 miles per hour through standard air. Consider the sides and top of the train as a smooth flat plate 30 feet wide. If the boundary layer is turbulent, what horsepower must be expended to overcome the skin-friction drag?

101. The main portion of a torpedo consists of a cylinder 21 inches in diameter and 18 feet long. Consider the skin friction the same as that of a flat plate of the same area. For a turbulent boundary layer, what power is required to overcome the skin friction if the torpedo moves 50 miles per hour through salt water (specific weight = 64.0 pounds per cubic foot)?

102. A horizontal wind at standard conditions flows over a horizontal flat square area 500 feet on each side. The air enters across one edge of the area with a uniform velocity of 30 miles per hour. What is the frictional force on this area if the boundary layer is turbulent?

103. A boat 90 feet long has a total wetted surface of 4000 square feet. Calculate the skin-friction drag at 10 miles per hour in fresh water for a turbulent boundary layer. Treat the wetted surface as a flat plate of the same length as the boat.

CHAPTER 11

Dynamic Lift and Propulsion

The preceding chapter discussed resistance, the component of the resultant force in the direction of the undisturbed flow. The present chapter deals with the other component, the lift, at right angles to the undisturbed flow. Note the generality of this definition of lift; the lift is not necessarily vertical. This chapter is limited to the steady flow of an incompressible fluid.

92. Rotating cylinder and Magnus effect

The "curve" of a baseball, a tennis ball, or a golf ball is a familiar example of the development of a dynamic lift; a twirl or spin of the ball is necessary in order to produce the curve. Serious attention was given to this phenomenon when it was observed that the trajectories of cannon balls showed a considerable lateral deflection from the vertical plane through the initial direction of the shots. Such deflection was found even in the absence of winds. This apparently strange displacement was associated with the rotation of the cannon balls.

Figures 103 and 104a show that the flow around a stationary cylinder is symmetrical with respect to a line through the cylinder center in the direction of the undisturbed flow. The pressure distribution on one side of this line is the same as that on the other side; therefore, there is no lift. If the cylinder in Fig. 103 were rotated clockwise in a viscous fluid, for example, the velocities above the cylinder would be higher than those below the cylinder. Application of the energy equation shows that the pressures on the lower surface would be greater than those on the upper surface—there would be a force at right angles to the undisturbed flow.

This effect of lift generation due to spinning is commonly called the Magnus effect after its discoverer Magnus. The Flettner rotor,[1] which employs the Magnus effect, has been applied to the propulsion of marine vessels. Vertical cylinders are extended some distance above the deck. Each cylinder is rotated about its axis by a small motor, and an air force is produced for moving the craft. It is to be noted that the unsymmetrical flow around the rotating cylinder is caused by viscosity effects. If the fluid were nonviscous, the rotating cylinder itself would not generate a lift.

The behavior of a frictionless fluid, however, is helpful in explaining certain features of the lift action. The two-dimensional unsymmetrical

[1] *The Story of the Rotor* by A. Flettner. F. O. Willhoft, New York, 1926.

flow of a frictionless fluid shown in Fig. 113 can be obtained by the superposition of two separate flows: (1) a translatory flow such as shown in Fig. 103; and (2) a circulatory flow, or a free vortex whose origin is at the center of the cylinder. The vector addition of these flows can be made by the methods outlined in Chapter 3. The streamlines in Fig. 113 have been so drawn that the volumetric rate of flow is the same in each space between the streamlines. Thus the spacing between streamlines gives an indication of the magnitude of the velocity. The streamlines are crowded (velocity is relatively high) over the top surface,

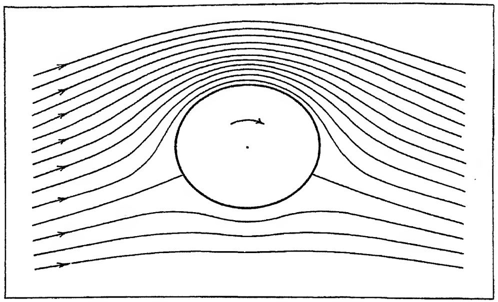


Fig. 113. Two-dimensional unsymmetrical flow around a circular cylinder.

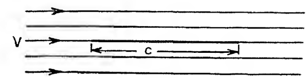
whereas the streamlines are spread apart (velocity is relatively low) over the lower surface.

93. Relation between lift and circulation for two-dimensional flow

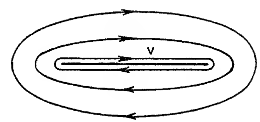
An elaborate, generalized, and rigorous analysis could be given of the important relation between lift, density, undisturbed velocity, and the so-called *circulation*. A simplified treatment, however, will be presented in order to bring out as directly as possible the fundamental physical features. Figure 114 shows the two-dimensional flow of a frictionless fluid around a flat plate. The chord is c, and the length perpendicular to the plane of the diagram is unity. For a pure rectilinear flow, Fig. 114a, there is no lift because the velocities, and therefore the pressures, are the same on both sides of the plate. Assume a clockwise circulatory flow around the plate as shown in Fig. 114b. Let v be the average

velocity along a closed curve adjacent to the plate. There is no lift because the velocities, and therefore the pressures, are the same on both sides of the plate.

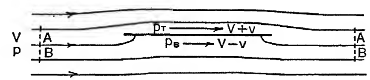
Figure 114c shows the results of superimposing the circulatory flow on the pure rectilinear flow. Over the top of the plate the average fluid velocity is $V + v$, whereas on the lower side the average velocity is



(a) Pure rectilinear flow.



(b) Pure circulatory flow.



(c) Combined rectilinear and circulatory flow.

Fig. 114. Two-dimensional flow around a flat plate.

$V - v$. Above the plate there is a high velocity and a resulting low pressure, whereas below the plate there is a low velocity and a high pressure. The combination of pressures produces an upward force or lift. Let p_T be the average pressure on the top side, and p_B be the average pressure on the bottom side. Application of the energy equation along the stream channel AA from the left to the top side, and application of

the energy equation along the stream channel BB from the left to the bottom side gives

$$p + \tfrac{1}{2}\rho V^2 = p_T + \tfrac{1}{2}\rho(V + v)^2,$$
$$p + \tfrac{1}{2}\rho V^2 = p_B + \tfrac{1}{2}\rho(V - v)^2.$$

Combining these equations show that

$$p_B - p_T = 2\rho V v.$$

The lift force L on the area is

$$L = (p_B - p_T)c = \rho V(2vc). \tag{98}$$

The product $2vc = \Gamma$ (Greek letter gamma) is called the *circulation*. Thus

$$L = \rho V \Gamma. \tag{99}$$

The lift per unit length equals the product of density, undisturbed velocity, and circulation. Equation (99) is frequently called the Kutta-Joukowsky relation, and might be regarded as an ideal or a theoretical maximum lift relation.

94. Circulation

Circulation is formally defined as the line integral of the velocity along a closed contour line in the fluid. This line integral is the integral of the product of the contour element ds and the component of the velocity in the direction of ds. Figure 115 shows a closed contour arbitrarily drawn in a field of flow. Let q be the velocity at any point, and θ the angle between q and the contour element ds. The circulation Γ around the closed contour is represented by the following integral:

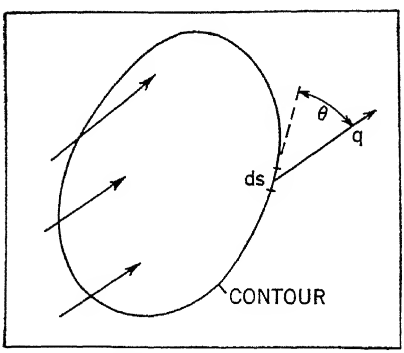


FIG. 115. Contour line in a field of flow.

$$\Gamma = \int_c q \cos\theta \, ds. \tag{100}$$

In the foregoing example of the flat plate, one contour length is $2c$, and the corresponding tangential velocity is v. Thus $\Gamma = 2vc$. A rotating cylinder in a mass of viscous fluid provides another example. Let a be the radius of the cylinder in Fig. 116, and V_P the peripheral velocity of the surface of the cylinder. A very thin layer of fluid adheres to the

surface of the cylinder; this layer of fluid has the velocity V_P. A free vortex is generated in the fluid. Let V be the tangential velocity at any radius r in the mass of moving fluid; thus $V_P a = Vr =$ the constant K. The surface of the cylinder will be selected as the contour for the calculation of the circulation. Then

$$\Gamma = 2\pi a V_P = 2\pi K. \quad (101)$$

Any other arbitrarily selected contour would give the same result. Thus, the circulation around the center of a free vortex is constant, and is independent of the contour selected. A loose analogy might be drawn between work done on a body and circulation. Force corresponds to velocity, and the body displacement corresponds to the contour element ds.

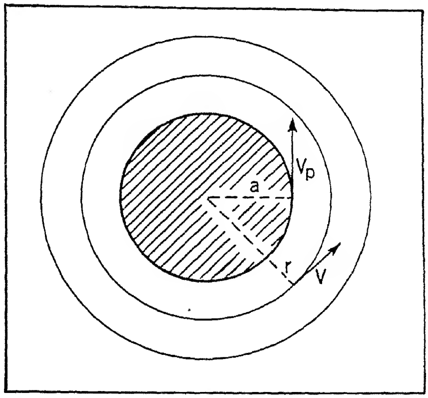


Fig. 116. Circulation around a rotating cylinder.

95. Generation of circulation

The Kutta-Joukowsky relation brings out the important fact that both a rectilinear flow and a circulation are necessary in order to produce a lift. The generation of circulation for a rotating ball or cylinder in a viscous fluid is not difficult to visualize. A very thin layer of the viscous fluid adheres to the surface or ball, and a free vortex or circulation is developed. The combination of circulatory flow and the rectilinear flow gives a resultant flow that generates a lift.

The development of lift for a foil, like an airfoil, a propeller blade element, a turbine blade element, or a pump vane element, however, requires further investigation. The flow is symmetrical about the section axis for the symmetrical section shown in Fig. 117; there is a rectilinear flow, but there is no circulation, and therefore no lift. Figure 118 shows a photograph of the two-dimensional flow around a section placed at an angle of attack with the undisturbed flow. The flow is from left to right. The streamlines were equally spaced some distance ahead of the nose. Consider first the flow over the top surface. The velocity is increased in a region near the nose (the streamlines are crowded). The velocity decreases in a region near the trailing edge; the flow resembles that in a diverging channel. Consider next the flow under the lower surface. The velocity is low below the nose. In a region near the

trailing edge the velocity increases; the flow resembles that in a converging channel. The velocity and pressure distribution is not uniform over each surface. The net effect, however, is a lift in the vertical direction.

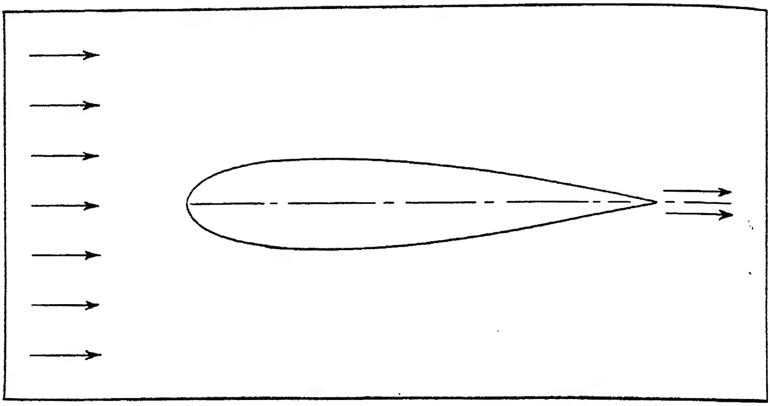


Fig. 117. A symmetrical section.

The development of circulation for a foil can be explained by referring to the phenomenon of a starting vortex. Two streams meet at the trailing edge in the symmetrical flow in Fig. 117; the lower stream has the same velocity as the upper stream. If the same section were placed at

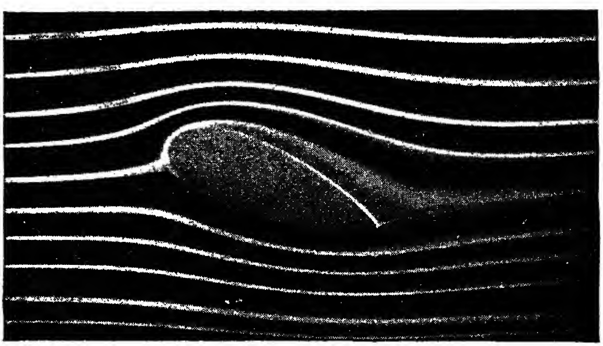


Fig. 118. Two-dimensional flow around a lifting vane.

an angle of attack with the stream, if the section were curved, or nonsymmetrical, then the two different streams at the trailing edge might meet with different velocities. In Fig. 119 the fluid from the lower side has a higher velocity at the trailing edge then the fluid from the upper

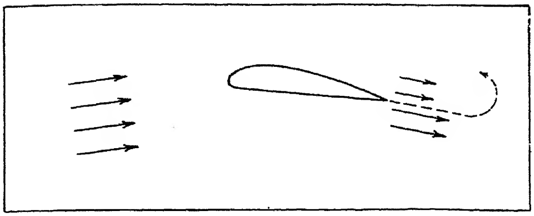


FIG. 119. Flow with different velocities at the trailing

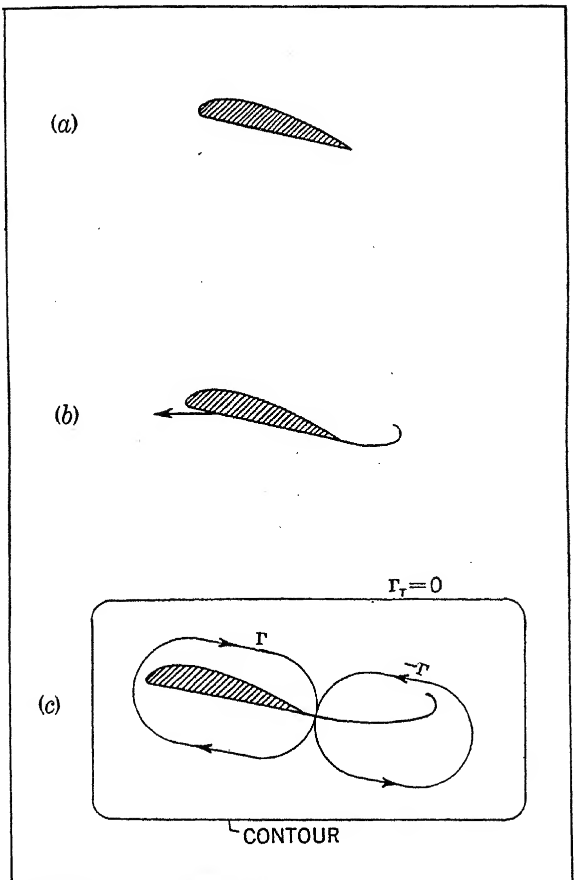


FIG. 120. Development of circulation around a lifting vane.

side. A surface of discontinuity is formed (velocity is discontinuous). If the fluid were nonviscous, each stream would not influence the other. For a real fluid, however, a vortex forms at the trailing edge.

Figure 120a shows a foil at rest in a stationary body of fluid. There is no circulation along any closed contour around the foil. A starting vortex begins to form as the section is moved. This formation is illustrated in Fig. 120b for movement of the foil towards the left. After some motion the vortex develops, breaks away from the body, and then moves downstream. The total circulation Γ_T around the contour is zero at all stages. A circulation around the foil is produced which is equal in magnitude, but opposite in direction, to that of the vortex which has shed. In symbols, $\Gamma_T = \Gamma - \Gamma = 0$.

The lift is vertical for the conditions shown in Fig. 120c. A starting vortex can be easily demonstrated by moving a foil through water having aluminum or some other powder on its surface, or simply by moving a spoon in a cup of coffee.

96. Lift coefficient

It is common practice to express the lift in the form

$$L = \text{lift} = C_L \frac{\rho V^2}{2} \text{ (area)}, \tag{102}$$

where C_L is a dimensionless lift coefficient. Figure 121 shows some conventions frequently employed. The geometric chord of length c is an arbitrary line usually established by the designer in laying out the section. The angle α between the chord and the line of the undisturbed velocity V is called the angle of attack. The area in Equation (102) is commonly taken as the projected chord area.

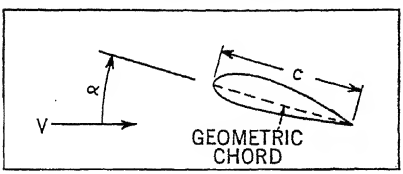


FIG. 121. Notation for a lifting vane.

Numerous lifting-vane sections, particularly symmetrical sections, show a behavior which in many respects approaches that of a flat plate. The lift coefficient for a flat plate might be regarded as a theoretical maximum or ideal. Kármán[2] gives the following theoretical value for the lift coefficient of a thin flat plate in two-dimensional flow:

$$C_L = 2\pi \sin \alpha. \tag{103}$$

[2] *General Aerodynamic Theory—Perfect Fluids* by T. von Kármán, vol. II, Division E, of *Aerodynamic Theory* edited by W. F. Durand. Julius Springer, Berlin, 1935.

Experimental results for modern airfoil sections, in a normal range of operation, show lift coefficient values about 90 per cent of the foregoing theoretical value.

Measured values of C_L for different sections can be found in such extensive reports as those of the N.A.C.A. Some typical features are illustrated by the sample shown in Fig. 122. The lift coefficient increases, reaches a maximum, and then drops as the angle of attack increases. The *stall* is the condition of a lifting vane (such as a pump vane, propeller

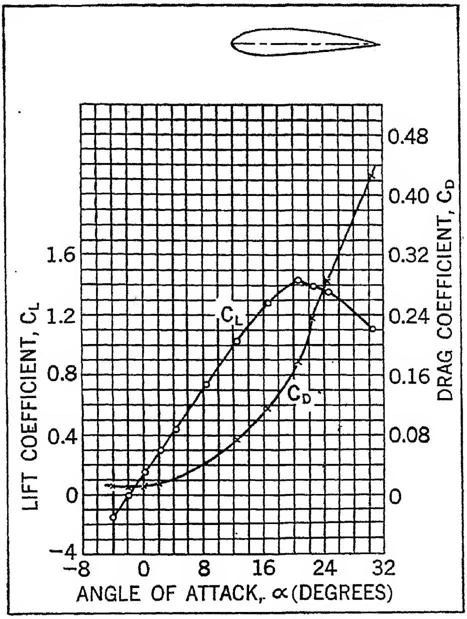


FIG. 122. Lift and drag coefficients plotted against angle of attack for N.A.C.A. 2418 airfoil. Reynolds' number = 3,060,000. (From N.A.C.A. Technical Report No. 669.)

element, airfoil, or airplane) in which it is operating at an angle of attack greater than the angle of attack at maximum lift. At the stall the fluid separates from the vane, and forms a marked eddying wake. Figure 122 shows that the drag coefficient increases considerably as the stall is reached.

97. Separation

The flow associated with the stall of a lifting vane occurs in many applications in which a fluid moves in a diverging channel. The flow

in various fluid machines (centrifugal pumps, turbines, fans, and the like) and in fluid meters (such as venturis) offers common examples. Figure 123 indicates a converging channel. If the fluid is incompressible, the fluid is accelerated in the converging section; some pressure head is converted into velocity head. In general, this conversion is a stable and efficient process; the energy losses are small, and there is no eddy formation.

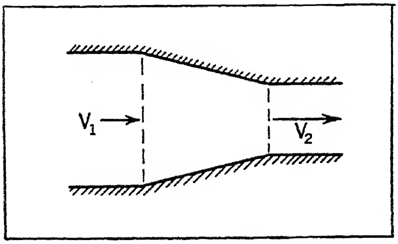


FIG. 123. Converging channel.

The flow in a diverging channel, on the other hand, may be unstable if the angle of divergence is appreciable. This flow involves a conversion from velocity head to pressure head. This process is much more difficult and troublesome than the reverse process. In the diverging channel some of the kinetic energy is dissipated into unavailable thermal energy because of the viscosity of the fluid; a full conversion or restoration into pressure head may not be realized. The fluid may not completely fill the channel, and separation may take place, just as in the stall of a lifting vane. Flow in a diverging channel, then, is a relatively unstable, inefficient process, and may be accompanied by large energy losses and eddy formation.

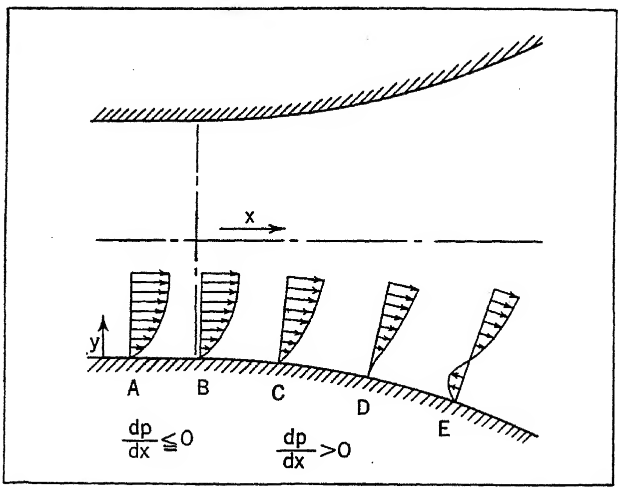


FIG. 124. Boundary layers in a diverging channel or diffuser.

Figure 124 shows the boundary layers at successive points along the wall in a diverging channel or diffuser. The ordinate y perpendicular to the wall has been exaggerated for the sake of illustration. At section B there is a certain velocity distribution in the boundary layer. A boundary layer with flow toward the right could be maintained if all the kinetic energy were available for conversion. Some kinetic energy is dissipated into unavailable thermal energy, however, with the result that the velocity in the boundary layer is *less* than what it would be without friction. At C the flow velocity is reduced. At D the velocity distribution curve is normal to the wall; D is the point of separation. At E there is a reversed flow in the boundary layer; an eddy is formed. The formation of eddies represents further dissipation of available energy into unavailable energy.

Let p be the static pressure, and x the distance taken positive to the right along the channel in Fig. 124. The ratio dp/dx is the pressure gradient. In a straight channel dp/dx would be zero if no friction were present; dp/dx would be negative (< 0) if friction existed. In a converging channel dp/dx would be negative. In the diffuser section dp/dx is positive (> 0). The danger of separation can be reduced by reducing dp/dx, as by making the angle of divergence small.

98. General features of blade screws

Among the numerous applications involving lifting vanes are those which might be classified as blade screws. A blade screw may be defined as any device with radial blades which rotates about its axis during its motion relative to the fluid. Four principal types are: (1) *Propeller.* Air and water craft, for example, are usually driven by one or more propellers whose axial thrust overcomes the drag of the craft. The propeller is rotated by a torque from some suitable power unit. (2) *Windmill.* This blade screw is moved by a current of fluid, in the direction of its axis, which causes the blade screw to rotate. The blade rotation develops a torque which may be used for the generation of power. (3) *Fan.* This familiar blade screw is used for producing a current of fluid. (4) *Anemometer.* This device is sometimes used as a meter, to determine the relative axial velocity by a measurement of the rate of rotation.

There are certain general features of each type which are similar. It is convenient to discuss these features by specific reference to propeller action. Let n be the rotational speed of the blade screw, in revolutions per unit time. The peripheral velocity (in the plane of rotation) of the element at A in Fig. 125 is $V_P = \pi nd$. Let V_0 be the velocity of the propeller along its axis, or the forward velocity of the craft to which the propeller is attached. Figure 126 shows the velocity and force components at a blade element. This blade element follows a helical path

in space. The resultant velocity V_R is the vector sum of V_0 and V_P. The resultant force R acting on the blade element can be broken up into two components, a lift L perpendicular to V_R, and a drag parallel to V_R.

Propeller study is concerned with two other components of R. One component of R is T, a thrust force along the axis of rotation for moving the craft; T does useful work. The other component of R, parallel to the plane of rotation, is the torque force F produced by the power unit.

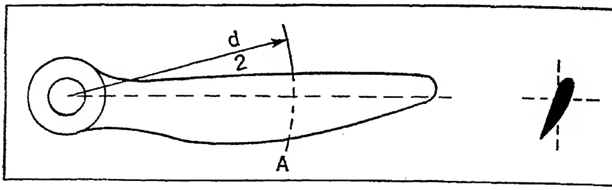


FIG. 125. Blade of a propeller.

α is the angle of attack of the lifting-vane element, and θ is the blade angle at the section. $\alpha = \theta - \tan^{-1}(V_0/V_P)$. Since V_P varies with the blade radius, the blade angle changes with radius; the blade is twisted.

Let D be the overall diameter of the propeller. The ratio V_0/n equals the distance the blade element advances during one revolution, and is called the advance per turn or the effective pitch. The dimensionless ratio V_0/nD is called the advance-diameter ratio of the screw, and is

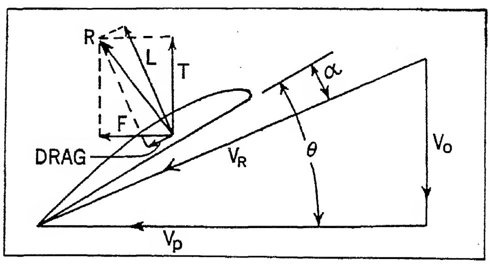


FIG. 126. Forces and velocity components at a blade element of a propeller.

useful in correlating data. The geometrical pitch of a propeller element is the distance the element would advance along its axis of rotation if it were moving along a helix of slope equal to its blade angle. The *nominal* or *standard* geometrical pitch of a whole propeller is frequently taken arbitrarily as the pitch of the section at two-thirds of the radius.

The analysis of the performance of a blade screw based on an integration of the behavior of the individual elements is frequently referred to as the blade-element theory. The interested student will find full discus-

sions in such works as those of Wieck and Glauert (see references at end of chapter).

99. Momentum and energy analysis of propellers

A momentum and energy study of an idealized blade screw is useful in giving some general idea of the main phenomena, and in giving values of maximum or ideal performance. It will be assumed that all fluid elements passing through the screw disk have their pressures raised by exactly the same amount. The rotational motion actually imparted to the fluid will be neglected.

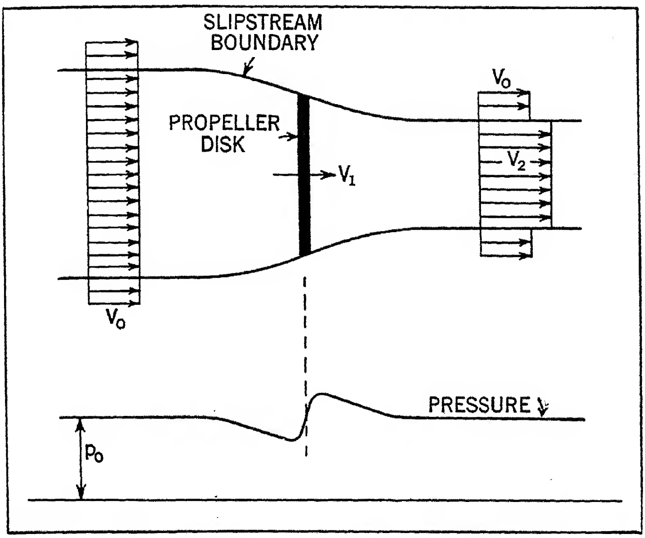


FIG. 127. Propeller slipstream.

A propeller, as on air or marine craft, moves with the craft velocity V_0. The same relative motion between fluid and propeller exists if the blade screw rotates in a fixed plane and the fluid approaches some distance ahead with the velocity V_0, as shown in Fig. 127. The stream of fluid passing through the propeller disk is called the *slipstream* or the *propeller race*. The fluid velocity increases as it approaches and leaves the propeller. At the propeller the velocity is V_1; at some distance behind the propeller the velocity is V_2 and the pressure is p_0. As indicated diagrammatically in the lower portion of Fig. 127, the pressure decreases from the value p_0, rises at the blade, and then drops to the value p_0. Momentum has been added to the fluid by the axial thrust T of the propeller.

The mass of fluid passing through any section per unit time is $(\pi/4)D^2\rho V_1$. Since force equals the time rate of change of linear momentum,

$$T = \frac{\pi D^2}{4} \rho V_1(V_2 - V_0). \tag{104}$$

The relation between the velocities can be found by applying the general energy equation. Consider an accounting for unit weight of fluid flowing in the slipstream. Then

$$\frac{p_0}{w} + \frac{V_0^2}{2g} + \text{work} = \frac{p_0}{w} + \frac{V_2^2}{2g}, \tag{105}$$

where the "work" term represents the work done on unit weight of fluid by the propeller. The total work done on the fluid per unit time is TV_1. The work done per unit weight is

$$\frac{TV_1}{\frac{\pi D^2}{4} V_1 w}.$$

The general energy equation then becomes

$$\frac{V_0^2}{2g} + \frac{4T}{\pi D^2 w} = \frac{V_2^2}{2g} \qquad \text{or} \qquad T = \frac{\pi D^2}{8} \rho(V_2^2 - V_0^2). \tag{106}$$

A comparison of Equations (104) and (106) shows that V_1 is the arithmetic mean of V_0 and V_2:

$$V_1 = \frac{V_0 + V_2}{2}. \tag{107}$$

The useful power output, in driving the craft, is TV_0. The ideal efficiency η (Greek letter eta) is defined as the ratio of power output to power input, or

$$\text{ideal efficiency} = \eta = \frac{TV_0}{TV_1} = \frac{V_0}{V_1}. \tag{108}$$

The actual propeller efficiencies obtained throughout the working range for good designs are usually from 80 to 88 per cent of the ideal efficiency. Equations (104), (107), (108) can be arranged to obtain the following relation:

$$\eta = \frac{1}{1 + \frac{2T}{\pi D^2 \rho V_1 V_0}}. \tag{109}$$

Equation (109) shows that the ideal efficiency increases with: (a) increase in propeller diameter, (b) increase in density, (c) increase in forward velocity, and (d) decrease in thrust. One practical conclusion shows that the propeller diameter should be as large as possible.

One limit on the diameter of propellers operating in air is set by the effect of shock waves, which occur when the propeller tip speed equals or exceeds the velocity of sound. This shock-wave effect will be discussed in the next chapter. One limit on the diameter of propellers operating in water is set by *cavitation.* Cavitation is a phenomenon in which the fluid pressure equals the vapor-pressure at the existing temperature. Vapor bubbles alternately form and collapse; this action may cause serious pitting of the blade screw, and a marked decrease in efficiency. Cavitation requires serious consideration in connection with marine propellers. Cavitation is discussed in Chapter 17.

100. Momentum analysis of finite lifting vane

The general features of the action of a finite lifting vane can be brought out by an elementary momentum analysis. It is assumed that the lifting

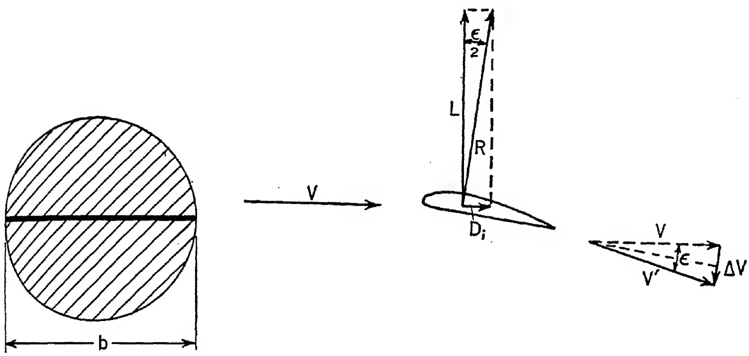


FIG. 128. Lifting vane of finite span.

vane in Fig. 128 deflects a cylindrical stream of fluid having a diameter equal to the span b of the vane. The fluid approaches the lifting vane with the velocity V. The velocity after leaving the vane is V'. The vane changes the velocity of the fluid by the amount ΔV. The reaction, in the opposite direction of ΔV, is the total force R of the fluid acting on the vane. The angle ϵ (Greek letter epsilon) is commonly called the *downwash angle.* If $V' = V$ in magnitude, than $\Delta V = V\epsilon$. R is resolved into two components, a lift L and a drag D_i. D_i is commonly called an *induced drag* because it is a drag *induced* by the lift. Since $R = M\Delta V$, where M is the mass of fluid deflected per unit time,

$$R = \frac{\pi b^2}{4} \rho V^2 \epsilon. \tag{110}$$

L is approximately equal to R for small angles of ϵ. With the use of the lift coefficient C_L, and the area S of the foil, there results:

$$L = \frac{\pi b^2}{4} \rho V^2 \epsilon = C_L \rho \frac{V^2}{2} S, \tag{111}$$

$$C_L = \frac{\pi b^2}{2S} \epsilon.$$

D_i can be written as

$$D_i = R \frac{\epsilon}{2} = \frac{\pi b^2}{4} \rho V^2 \frac{\epsilon^2}{2} = C_{D_i} \frac{\rho V^2}{2} S;$$

or

$$C_{D_i} = \frac{\pi b^2}{4} \frac{\epsilon^2}{S}, \tag{112}$$

where C_{D_i} is an induced drag coefficient.

Eliminating ϵ from Equations (111) and (112) gives

$$C_{D_i} = \frac{C_L^2}{\pi \left(\frac{b^2}{S}\right)}, \tag{113}$$

where the ratio b^2/S is called the *aspect ratio*. For a rectangular foil with chord c, the aspect ratio is simply b/c. Comparison of Equation (113) with observed values indicates that it is a reasonable first approximation. A closer agreement with experimental data is reached if the cross-sectional area of the deflected stream is taken as an ellipse instead of a circle.

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PROBLEMS

104. An infinitely long cylinder, 2 feet in diameter, is in an air stream, standard conditions, moving at 5 miles per hour. The cylinder is rotated at 300 revolutions per minute. The actual lift is 2.9 pounds per foot of length. What is the ratio between the actual and ideal lifts?

105. An airfoil having a span of 40 feet moves through standard air at 180 miles per hour. At the middle of the airfoil the circulation has a maximum value of 200 feet squared per second. Assume two-dimensional flow. Because of the shape of the foil, the circulation decreases symmetrically from the maximum at the center to zero at each tip. The circulation can be represented by the ordinates of a semiellipse, with one axis equal to the span. What is the total lift?

106. Show that the circulation around a forced vortex is a function of the angular speed and the radius.

107. A bird rests on a perch in a cage. The cage is placed on weighing scales. The bird leaves the perch, and flies horizontally in the cage. For steady flight, is there any change in the scale reading?

108. A board, having an area of 6 square feet, is towed under water at a speed of 7 miles per hour. The angle of attack is 3°. What is the theoretical or ideal lift?

109. A helicopter weighing 500 pounds is hovering at a certain level in standard air. The diameter of the propeller is 10 feet. What is the effective or useful work done? What is the power delivered by the propeller to the fluid?

110. A propeller of 7-inch diameter drives a water craft at a uniform velocity. The propeller develops a thrust of 20 pounds, has an ideal efficiency of 80 per cent, and rotates at 1200 revolutions per minute. What is the speed of the craft? If the angle of attack at 75 per cent of the radius is 3°, what is the blade angle at this radius?

111. An airplane propeller 8 feet in diameter develops a thrust of 1400 pounds when flying at 200 miles per hour. What is the ideal propeller efficiency? What is the theoretical value of the power absorbed by the propeller?

112. A propeller-type fan of 12-inch diameter operates in standard air. Some distance ahead of the fan, on the discharge side, the velocity is 50 feet per second. The ideal propeller efficiency is 80 per cent. What power is delivered by the driving motor?

113. A propeller-type wind mill, 9 feet in diameter, operates in a wind of 30 miles per hour. Theoretical propeller efficiency is 80 per cent. Calculate the power absorbed by the propeller.

114. Starting with Equations (104), (107), and (108), derive Equation (109).

CHAPTER 12

Dynamics of Compressible Flow

. . The fundamental distinction between drag at supersonic and subsonic speeds has been pointed out by quite a number of physicists and ballisticians dealing with the problem; namely, that the energy loss corresponding to the drag at supersonic speeds is dissipated in the waves accompanying the projectile, in particular in the head wave emanating from the nose of the bullet, whereas the energy loss at subsonic velocities is mainly dissipated in eddies produced in the rear of the body.—T. VON KÁRMÁN.[1]

Density variations may have an appreciable effect on the flow; such variations may occur with both liquids and gases. The flow of gases through various machines (internal combustion engines and compressors, for example), the motion of craft or propeller sections at high velocities, the motion of projectiles, the high-velocity flow through nozzles and orifices, and sound phenomena (acoustics) are some examples of many in which the compressibility of a fluid exerts an influence. The purpose of the present chapter and the next is to provide some introduction to compressibility effects of practical importance.

101. Bulk modulus

The reciprocal of the modulus of elasticity or *bulk modulus* of a fluid is a direct measure of the compressibility of the fluid. If a volume of fluid v under a pressure p were subjected to an increase in pressure dp, there would be a decrease in volume dv. The bulk modulus E is defined as the ratio

$$\text{bulk modulus} \doteq E = -\frac{dp}{dv/v}. \tag{114}$$

The dimensionless ratio dv/v is a volumetric strain. The bulk modulus has the dimension of force per unit area. The negative sign in Equation (114) signifies a decrease in volume for a positive increment in pressure. E is about 300,000 pounds per square inch for water at ordinary conditions; water is about 100 times as compressible as mild steel. The bulk modulus for a gas depends upon the particular pressure-volume relation followed during the compression or expansion process. For an isothermal process, pv = constant, and

[1] *Problems of Flow in Compressible Fluids*, in the book *Fluid Mechanics and Statistical Methods in Engineering*. University of Pennsylvania Press, Philadelphia, 1941.

$$E = -\frac{dp}{dv/v} = \frac{dp}{dp/p} = p. \tag{115}$$

The bulk modulus equals the pressure. For an adiabatic process, pv^k = constant, and

$$E = -\frac{dp}{dv/v} = \frac{dp}{dp/kp} = kp. \tag{116}$$

The bulk modulus equals the product of the pressure and the ratio of the specific heats. Some values of k are listed in Table 2, page 5. Most liquids have a relatively high bulk modulus, whereas the bulk modulus for a gas is usually relatively low.

102. Velocity of pressure propagation

In a perfectly rigid and incompressible medium an impulse is transmitted instantaneously from one element to another. In an elastic or

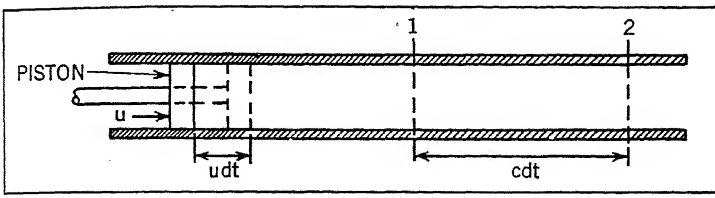

FIG. 129. Notation for calculating velocity of pressure propagation.

compressible medium, however, the transmission of a wave is retarded by the inertia of the displaced elements. The time of travel of a wave may be short or difficult to observe if the distance of travel is short; on the other hand, the time of travel may be noticeable if the distance is large. Thunder, for example, is heard some time after the lightning is seen if the observer is some distance away from the storm; a finite time is required for the sound wave to travel a certain distance through the air. Incidentally, it is sometimes recommended that one should not worry about the lightning if he hears the thunder. There are numerous applications, such as one method for measuring depth in the ocean, the sound detection of a submarine, and the location of a gun by sound ranging, which depends upon the fact that a pressure wave in a fluid requires time to travel a certain distance.

Figure 129 indicates a body of fluid at rest in a pipe; there is a piston or diaphragm at one end. The simplest case will be taken, that of motion in one direction along the axis of the pipe; such motion can be visualized by regarding the pipe as rigid in comparison with the fluid. The piston is suddenly moved with a velocity u during a time interval dt. This piston motion sends a pressure wave along the pipe with the constant velocity c; c is commonly called the velocity of sound or the acoustic

velocity. It will be assumed that the pressure increment dp is small in comparison with the absolute pressure p (rigorously speaking, the pressure change is infinitesimal). The pressure wave covers a distance cdt while the piston moves a distance udt. Let A represent the area of the piston, and $d\rho$ the density change. The mass of fluid displaced by the piston ($A\rho udt$) equals the gain in mass of the fluid between sections 1 and 2 which is ($Acdtd\rho$). Therefore,

$$A\rho udt = Acdtd\rho \qquad \text{or} \qquad c = u\frac{\rho}{d\rho} \tag{117}$$

Also, the force (Adp) exerted on the mass ($Acdt\rho$) equals the time rate of change in linear momentum. For a velocity change u,

$$Adp = Acdt\rho u \qquad c = \frac{}{\rho u} \tag{118}$$

Combining Equations (117) and (118) shows that $c^2 = dp/d\rho$.

$$\frac{dp}{dv/v} = \frac{dp}{d\rho/\rho},$$

$$= \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{E}{\rho}}. \tag{119}$$

c can be calculated by Equation (119) if E and ρ are known. For pressure waves in a gas, it is customary to regard the process as adiabatic; the pressure changes are quite rapid. For an adiabatic process in a gas,

$$c = \sqrt{\frac{kp}{\rho}}. \tag{120}$$

Equation (120) shows fair agreement with measured values. c is about 1120 feet per second for air at standard conditions. Using the relation $pv = RT$ gives

$$c = \sqrt{kgRT}. \tag{121}$$

Equation (121) shows that the velocity of pressure propagation in a gas varies as the square root of the absolute temperature. A pressure wave in the exhaust pipe of an internal-combustion engine, for example, travels at a higher speed than the pressure wave in the intake pipe.

103. Dynamic similarity for compressible flow—Mach's number

It was pointed out in Chapter 6 that various laws of similitude could be devised, depending upon the type of forces acting. Reynolds' number is proportional to the ratio $\frac{\text{inertia force}}{\text{viscous force}}$. Preceding chapters have shown how Reynolds' number is useful in problems of incompressible flow, in establishing dynamic similarity, as a criterion for type of flow, and as a

dimensionless parameter in correlating data. A significant dimensionless ratio for compressible flow is Mach's number.

Consider the flow of a compressible fluid around two geometrically similar bodies (or through two geometrically similar channels) in which the predominating forces are inertia, pressure, and elastic. If only three forces are involved, specifying two of the forces automatically specifies the third force because the three forces are in equilibrium (recalling d'Alembert's principle). A significant ratio is the ratio $\frac{\text{inertia force}}{\text{elastic force}}$. The inertia force is proportional to $\rho l^3(V/t)$ or ρl^2V^2, where l is some characteristic length or dimension. The elastic or compressibility force is proportional to El^2. Then $\frac{\text{inertia force}}{\text{elastic force}}$ is proportional to

$$\frac{\rho l^2V^2}{El^2} \qquad \frac{\rho V^2}{E} \qquad \frac{V^2}{c^2}.$$

If inertia and elastic forces determine the flow for a prototype, then mechanical similarity between model and prototype is realized when the ratio V^2/c^2 for the model equals the corresponding ratio V^2/c^2 for the prototype. The ratio V/c could be used just as well as the ratio V^2/c^2. The ratio $N_M = V/c$ is commonly called Mach's number. Subsequent discussions will bring out some of the important features characterized by Mach's number. A critical value is $N_M = 1$: this value marks a distinction between two different types of flow.

104. Some thermodynamic relations for gases

Some useful thermodynamic relations will be reviewed first before proceeding to a more detailed investigation of compressible flow. In making an energy balance in compressible-flow problems, there is the question of a convenient evaluation of the change in internal energy or stored thermal energy. The change in internal energy $(u_2 - u_1)$ depends solely on the initial and final states, and not on how these states were reached. Internal energy, like temperature and height, is a "point" function. Any path or process could be employed to evaluate $u_2 - u_1$. A nonflow process will be selected because it is convenient.

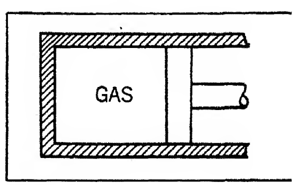

FIG. 130. Movable piston in a cylinder.

Imagine the nonflow process for a gas in a cylinder with a movable piston, as shown in Fig. 130. The heat added to the gas equals the gain

in internal energy plus the work done by the gas on the piston, that is,

$$q = u_2 - u_1 + \tfrac{1}{778} \int_1^2 p\,dv,$$

where the subscripts 1 and 2 refer to initial and final states respectively, and the relation is written for unit weight. The work done is zero $(dv = 0)$ for a constant-volume process. The heat added can be expressed as the product of specific heat and the change in temperature. Then, for a constant-volume process,

$$q = u_2 - u_1 = c_v(T_2 - T_1), \tag{122}$$

where c_v is the specific heat at constant volume. Since internal energy depends solely on temperature, the change in internal energy for *any* process equals $c_v(T_2 - T_1)$. Accordingly, for *any* nonflow process of a gas,

$$q = c_v(T_2 - T_1) + \tfrac{1}{778} \int_1^2 p\,dv. \tag{123}$$

For a constant-pressure process, $q = c_p(T_2 - T_1)$, where c_p is the specific heat at constant pressure, and the nonflow energy equation becomes

$$c_p(T_2 - T_1) = c_v(T_2 - T_1) + \frac{p(v_2 - v_1)}{778}.$$

Since $pv = RT$, and $k = \dfrac{c_p}{c_v}$

$$c_p = c_v + \frac{R}{778}; \tag{124}$$

$$c_v = \frac{1}{778}\left(\frac{R}{k - 1}\right). \tag{125}$$

105. Energy equation for compressible flow

The general energy equation, as presented in Chapter 4, is

$$q + \frac{p_1v_1}{778} - \frac{p_2v_2}{778} + \frac{\text{work}}{778} = u_2 - u_1 + \frac{V_2^2 - V_1^2}{2g(778)} + \frac{z_2 - z_1}{778}.$$

For many practical cases it is sufficiently accurate to regard the process as adiabatic. Consider the case in which $q = 0$, work $= 0$, and $z_2 - z_1 = 0$. Then

$$\frac{p_1v_1}{778} - \frac{p_2v_2}{778} = u_2 - u_1 + \frac{V_2^2 - V_1^2}{2g(778)}. \tag{126}$$

Each term in Equation (126) is expressed in British thermal units per pound of fluid flowing. Equation (126) applies to any fluid substance; the fluid may be a liquid, gas, or vapor. If the fluid is a gas following the

relation pv RT, then the energy equation becomes

$$\frac{V_2^2 - V_1^2}{2g(778)} = \frac{p_1 v_1}{778} - \frac{p_2 v_2}{778} + c_v(T_1 - T_2). \tag{127}$$

Using Equation (125), we can write Equation (127) as

$$\frac{V_2^2 - V_1^2}{2g} = \frac{k}{k-1}(p_1 v_1 - p_2 v_2). \tag{128}$$

Equation (128) is a convenient form of the energy equation for gas flow. Note that the process has not yet been fully specified. If the process is frictionless and reversible, then the gas follows the relation pv^k = constant. Since

$$\frac{p_2}{p_1} = \left(\frac{v_1}{v_2}\right)^k = \left(\frac{\rho_2}{\rho_1}\right)^k,$$

the energy equation becomes

$$\frac{V_2^2 - V_1^2}{2} = \left(\frac{k}{k-1}\right)\frac{p_1}{\rho_1}\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\right]. \tag{129}$$

Equation (129) has wide application to problems involving the flow of gas between streamlines, through orifices, through nozzles, and flow meters. Substituting the relation $c_1^2 = kp_1/\rho_1$ in Equation (129) gives

$$\frac{V_2^2 - V_1^2}{2} = \frac{c_1^2}{k-1}\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\right]. \tag{130}$$

or

$$\frac{p_2}{p_1} = \left[1 - \frac{k-1}{c_1^2}\left(\frac{V_2^2 - V_1^2}{2}\right)\right]^{\frac{k}{k-1}}. \tag{131}$$

106. Pressure at a stagnation point

The determination of the pressure at a stagnation point illustrates one application of the foregoing relations. Consider the flow of a frictionless fluid around a body, as indicated in Fig. 131. The pressure is p_0 and the velocity is V_0 in the undisturbed stream to the left. At the *stagnation point* S the fluid velocity is zero, and the stagnation pressure is p_S. The energy equation for an adiabatic process will be applied to the small stream-tube along the streamline 0 to S.

For incompressible flow, Equation (33) gives the relation

$$p_S = p_0 + \tfrac{1}{2}\rho V_0^2. \tag{132}$$

For the compressible flow of a gas, Equation (131) gives the relation

$$\frac{p_S}{p_0} = \left[1 + \left(\frac{k-1}{2}\right)\left(\frac{V_0^2}{c_0^2}\right)\right]^{\frac{k}{k-1}}, \tag{133}$$

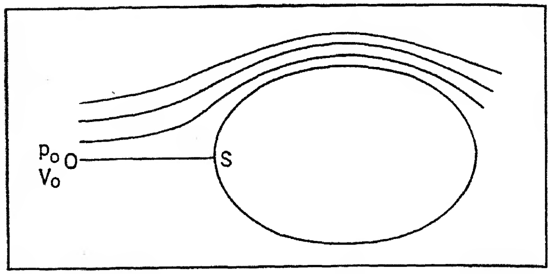

FIG. 131. Body in a stream of fluid.

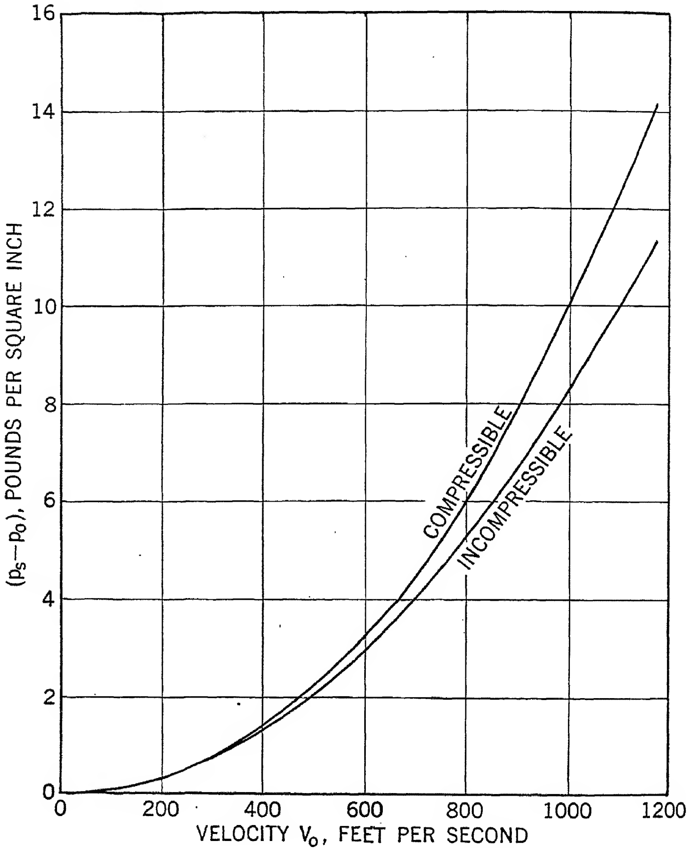

FIG. 132. Pressure at stagnation point plotted against undisturbed velocity for compressible and incompressible flow of air.

where c_0 is the velocity of pressure propagation in the undisturbed flow. Let $V_0/c_0 = N_M$. Then Equation (133) becomes

$$\frac{p_S}{p_0} = \left[1 + \left(\frac{k-1}{2}\right) N_M{}^2\right]^{\frac{k}{k-1}}. \tag{134}$$

If the term $\left(\frac{k-1}{2}\right) N_M{}^2$ is less than unity, then the right side of Equation (134) can be expanded in terms of a convergent power series. The result of this expansion and the substitution $\rho_0 = kp_0/c_0{}^2$ can be arranged in the form

$$p_S = p_0 + \frac{1}{2}\rho_0 V_0{}^2\left[1 + \frac{N_M{}^2}{4} + \left(\frac{2-k}{24}\right) N_M{}^4 \cdots\right] \tag{135}$$

The important feature is a comparison of Equations (132) and (135). Equation (135) shows that the stagnation pressure for compressible flow is higher than the stagnation pressure for incompressible flow. If N_M is very small, p_S for compressible flow may not be very much different from that for incompressible flow; the difference depends upon Mach's number in the undisturbed stream. Figure 132 shows graphically the variation of stagnation pressure with velocity for incompressible and compressible flow. The results shown in Fig. 132 are based on standard air in the undisturbed stream. Note the difference in pressures at high velocities.

107. Adiabatic flow through stream-tubes

An important difference between incompressible and compressible flow can be brought out by investigating the relation between area and velocity for a fluid flowing through a tube of varying cross-section. Let A represent the cross-sectional area of the tube at any section, ρ the density, and V the average velocity. The equation of continuity states

$$\rho VA = \text{constant}. \tag{136}$$

If the fluid is incompressible, ρ is constant, and the equation of continuity takes the familiar, special form

$$VA = \text{constant}. \tag{137}$$

Equation (137) is the relation for a hyperbola; there are no critical values in the area-velocity relation for incompressible flow. Differentiation of Equation (137) gives

$$\frac{dA}{A} = -\frac{dV}{V}. \tag{138}$$

The relation between area and velocity for an elastic or compressible fluid can be obtained by combining the equation of continuity, the equation of state, and the energy equation. The case for a gas following

an adiabatic process will be taken; thus the relations presented in Article 105 can be applied. Differentiating Equation (136) gives

$$\frac{dA}{A} = -\frac{dV}{V} - \frac{d\rho}{\rho}. \tag{139}$$

The energy Equation (128) can be expressed in the form

$$\frac{V^2}{2} + \left(\frac{k}{k-}\right. \quad \frac{p}{} = \text{constant.} \tag{140}$$

Differentiating Equation (140) gives

$$VdV + \left(\frac{k}{k-1}\right)\left[\frac{dp}{\rho} - \frac{pd\rho}{\rho^2}\right] = 0. \tag{141}$$

Equations (119) and (120) show that for a gas $c = \sqrt{dp/d\rho} = \sqrt{kp/\rho}$. Using Equations (119) and (120) in Equation (141) gives

$$VdV + \frac{c^2 d\rho}{\rho} = 0. \tag{142}$$

Combining Equations (139) and (142) gives the final, convenient form for compressible flow:

$$\frac{dA}{A} = -\frac{dV}{V}\left[1 - \frac{V^2}{c^2}\right] = -\frac{dV}{V}[1 - N_M{}^2], \tag{143}$$

where the ratio V/c is frequently called the *local* Mach's number.

The important feature is a comparison of Equations (138) and (143). The area-velocity relation for compressible flow differs from that for incompressible flow by the factor $1 - (V^2/c^2)$. Flow in which the fluid velocity is *greater* than the velocity of pressure propagation differs from flow in which the fluid velocity is *less* than the velocity of pressure propagation. When $N_M < 1$, dA/dV is negative; this sign indicates that the velocity increases in a converging channel and decreases in a diverging channel. The flow can be assumed incompressible if $N_M{}^2$ is very small in comparison with unity. When $N_M > 1$, dA/dV is positive; this sign indicates that the velocity *increases* in a diverging channel and *decreases* in a converging channel.

In order to integrate Equation (143) it is necessary to express the variable c in terms of V. Let c_0 be the velocity of pressure propagation for an initial state of rest ($V_0 = 0$). Then $c_0{}^2 = kp_0/\rho_0$. Using this relation and the relation $c^2 = kp/\rho$ in Equation (128) gives c in terms of V

$$c^2 = c_0{}^2 - \left(\frac{k-1}{2}\right)V^2. \tag{144}$$

Figure 133 shows the general variation of cross-sectional area A as a function of velocity V for an incompressible fluid and an elastic or compressible fluid. The curve marked "compressible" in Fig. 133 is the type of curve obtained by making the substitution

$$c^2 = c_0^2 - \left(\frac{k-1}{2}\right)V^2$$

in Equation (143) and then integrating.

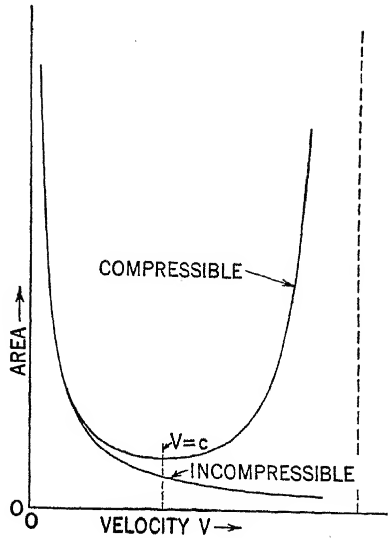

FIG. 133. Plot of cross-sectional area as a function of velocity for an incompressible fluid and a compressible fluid.

108. Action at subsonic and supersonic velocities

Under the conditions of Fig. 134, imagine an infinitesimal particle or a point disturbance moving through a fluid with a constant velocity V *less* than the velocity of sound c. A pressure wave is produced when the particle is at A. This pressure wave has a spherical front with a center at A. After a time interval t this wave front has traveled a distance ct. During this time interval the particle has moved a distance Vt, from A to B. The pressure wave reaches B before the particle; the pressure wave has been "telegraphed" ahead. The upstream fluid particles have some opportunity for adjustment to the motion before the body (or particle initially at A) reaches their positions. The intermediate points between A and B are sources of other spherical pressure waves. Since V is less than c, these other pressure waves will always be contained within the sphere of radius ct.

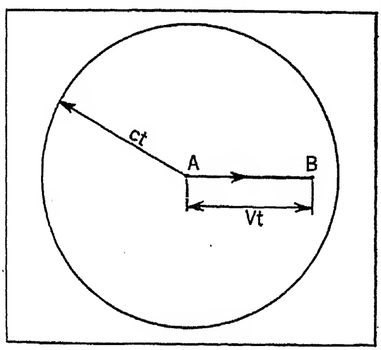

FIG. 134. Wave front produced by a particle moving at subsonic velocity.

Now turn to Fig. 135, and imagine a particle or point disturbance moving with a velocity V *greater* than the velocity of sound c. The

particle reaches point B before the pressure wave; the pressure wave is not moving fast enough to be telegraphed ahead of the particle. Intermediate points between A and B are sources of other spherical waves. The entire system of spherical pressure waves results in a conical front

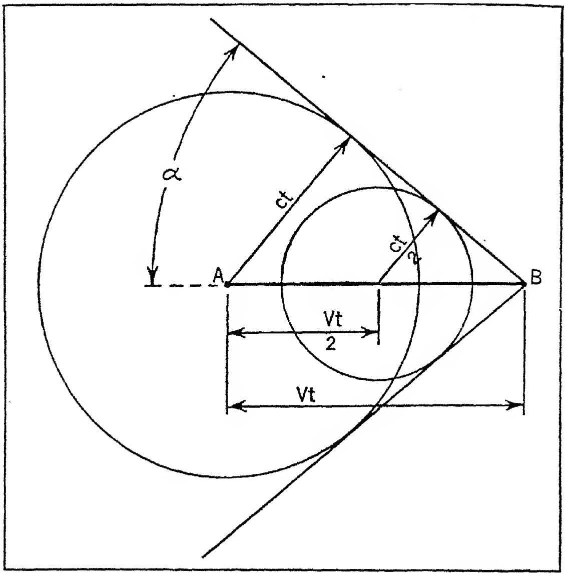

FIG. 135. Wave fronts produced by a particle moving at supersonic velocity.

with vertex at B. The half angle α at the cone vertex is sometimes called Mach's angle.

$$\alpha = \sin^{-1} \frac{ct}{Vt} = \sin^{-1} \frac{c}{V}. \tag{145}$$

When a body moves with a supersonic velocity, the pressure waves are behind the body. A considerable shock is produced as the body comes in contact with fluid particles initially ahead of it. The conical wave front, a form of discontinuity, is called a shock wave.

109. Photographic study of compressibility effects

Some ingenious arrangements of apparatus have been devised for photographing compressibility effects. The resulting photographs are variously called "shadow," "schlieren,"[2] "striae," or "streak" pictures.

[2] A discussion of the optical features of the schlieren method can be found in *Physical Optics* by R. W. Wood. Macmillan, New York, 1934.

Light from a suitable source is passed through the region of gas in motion. Small differences in density are made visible by means of differences in intensity of the light which has passed through the gas. The deflection or refraction of a light beam passing through a body of gas depends upon the density of the gas. Sudden changes in density result in definite bands of shadow or intense illumination. The sparkling or twinkling of stars is an example. The apparent direction and intensity of a star are

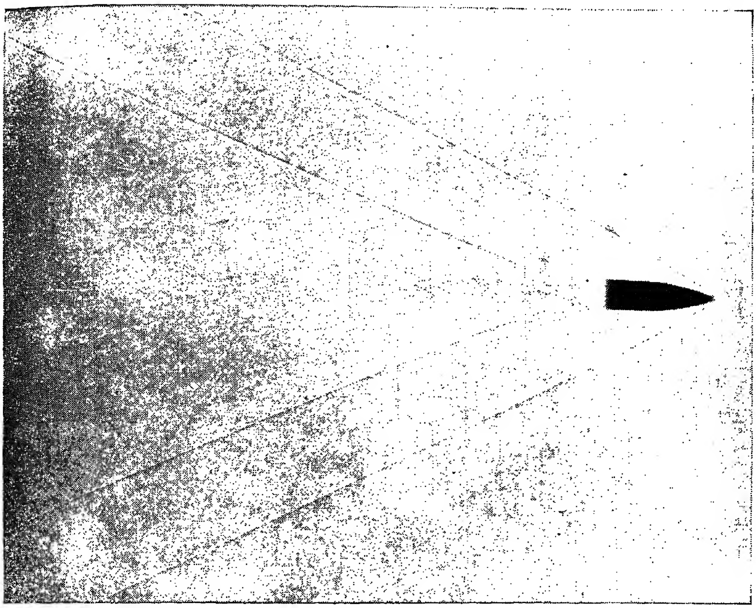

FIG. 136. Shadow picture of a 0.30-caliber service bullet.[3]

subject to rapid fluctuations as masses of air of varying density drift across the line of sight.

Figures 136 and 137 show shadow pictures of bullets taken at the National Bureau of Standards. The apparatus was so arranged that the illuminating spark occurred while the bullet was between the spark and the photographic plate. The bullet leaves an ordinary shadow; the shock waves and turbulent wave also give a shadow because of refraction effects. The bullet in each figure was probably moving at a speed of about 2700 feet per second. Each figure shows the head shock wave,

[3] Figures 136 and 137 were reproduced, with permission, from *Spark Photography and Its Applications to Some Problems in Ballistics* by P. P. Quayle, Scientific Paper No. 508 of the National Bureau of Standards, 1925, page 250.

the tail wave, and the eddies in the wake. Some investigators have claimed that the velocity of the bullet can be calculated with fair accuracy from the angle of the nose wave some distance from the bullet; V can be calculated by Equation (145) if α and c are known.

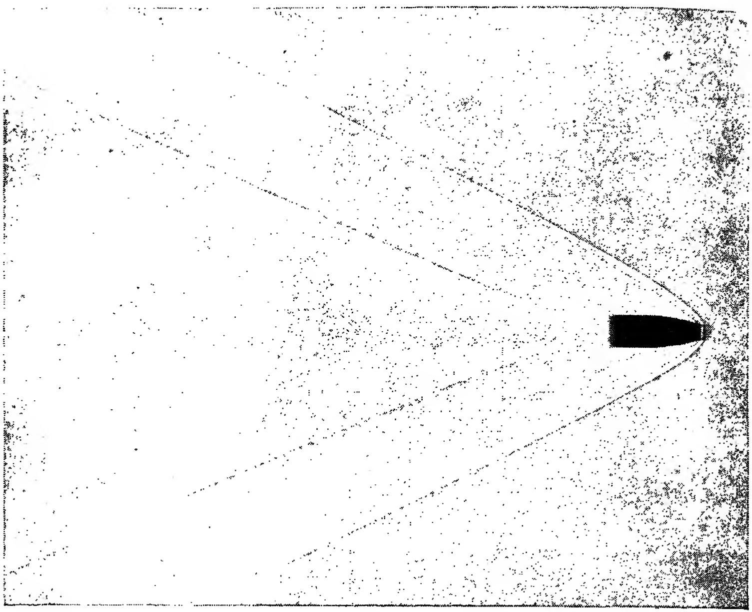

FIG. 137. Shadow picture of a 0.30-caliber service bullet with a modified head.[3]

110. Discharge through a converging tube or orifice

Several fundamental features of practical importance can be conveniently illustrated by specific reference to the flow of a gas through a nozzle, or a channel of varying cross section. Consider first the adiabatic frictionless flow in a short converging tube, as represented in Fig. 138. The case is taken in which the velocity V_1 at entrance is small in comparison with the velocity V_2 at the throat or section of minimum area. p_1 is the pressure at the entrance, and p_2 is the pressure at the throat. Then Equation (129) becomes

$$V_2 = \sqrt{\left(\frac{2k}{k-1}\right)\frac{p_1}{\rho_1}\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\right]}. \tag{146}$$

Let W represent the weight of gas passing through the tube per unit time, and A_2 the area at the throat. Then $W = A_2V_2/v_2$, where v is specific

[3] For footnote see page 171.

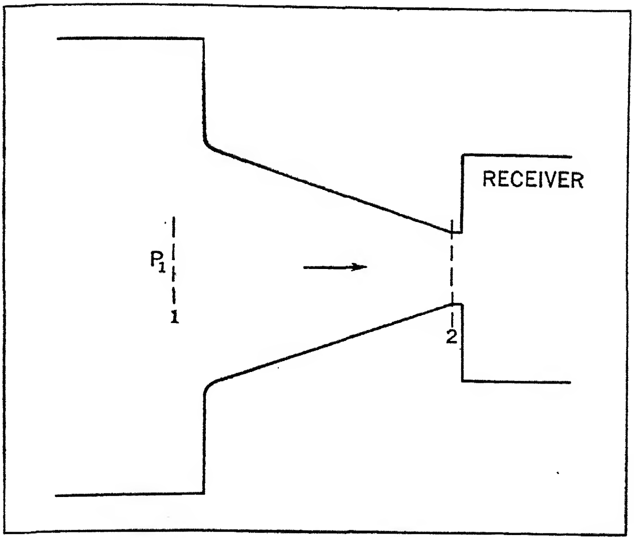

FIG. 138. Converging tube.

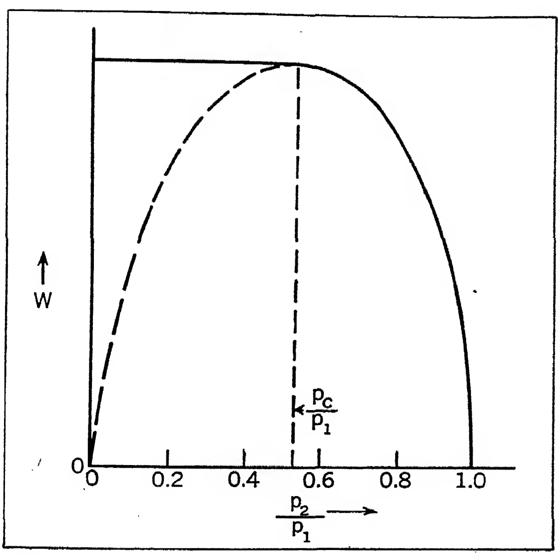

FIG. 139. Relation between weight rate of discharge and pressure ratio for a converging tube.

volume. Since $p_1v_1^k = p_2v_2^k$, W can be expressed as

$$W = A_2\sqrt{\left(\frac{2kg}{k-1}\right)\left(\frac{p_1}{v_1}\right)\left[\left(\frac{p_2}{p_1}\right)^{\frac{2}{k}} - \left(\frac{p_2}{p_1}\right)^{\frac{k+1}{k}}\right]}. \tag{147}$$

The variation of W with the ratio p_2/p_1, as given by Equation (147), is illustrated by the curved line (partly dotted and partly solid) in Fig. 139.

Figure 139 shows that W reaches a maximum value for a certain pressure ratio p_c/p_1. p_c will be called a *critical* pressure. The critical pressure can be determined by differentiating W with respect to p_2 and setting the result equal to zero. This operation gives

$$\frac{p_c}{p_1} = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}. \tag{148}$$

For air at normal conditions p_c/p_1 is about 0.53.

The dotted curve in Fig. 139 is not actually attained for flow *in* the *converging* tube. If the pressure p_2 is decreased from the value p_1 to p_c, the weight rate of discharge increases from zero to a maximum, as indicated by the solid curve in Fig. 139. Consider the conditions when the pressure p_2 equals p_c. From Equation (128),

$$\frac{V_c^2}{2g} = \left(\frac{k}{k-1}\right)(p_1v_1 - p_cv_c),$$

where V_c is the fluid velocity at the critical pressure, and v_c is the specific volume at the critical pressure. Then

$$V_c = \sqrt{\left(\frac{2gk}{k-1}\right)(p_1v_1 - p_cv_c)}. \tag{149}$$

Inasmuch as $p_1v_1^k = p_cv_c^k$, and $p_c = p_1\left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}$, Equation (149) becomes

$$V_c = \sqrt{\left(\frac{2gk}{k+1}\right)p_1v_1} = \sqrt{\frac{kp_c}{\rho_c}}, \tag{150}$$

where ρ_c is the density at the critical pressure. Equation (150) shows that the *fluid velocity at the throat equals the velocity of sound at the critical pressure*. Mach's number at the throat is unity.

If the pressure in the receiver is reduced below p_c, the pressure in the receiver cannot be telegraphed back into the throat of the nozzle because the fluid in the throat is moving with the velocity of pressure propagation. For receiver pressures less than p_c, the pressure *in* the throat is always p_c, and the weight rate of discharge always equals the maximum value.

On the other hand, if the pressure in the receiver is above p_c, then the fluid velocity at the throat is less than the velocity of sound; the receiver pressure can be telegraphed into the throat.

The above phenomena are illustrated by the flow in the exhaust port of an internal-combustion engine. If the cylinder pressure at the time of opening of the exhaust valve is so high that the critical pressure is above the pressure in the exhaust pipe, then the fluid velocity in the valve port equals the velocity of sound. For this condition a further reduction in the exhaust manifold pressure would not influence the flow rate. As exhaust proceeds, the cylinder pressure drops and the port velocity decreases below the velocity of sound.

A nozzle or orifice in which the critical pressure is reached in the minimum cross section is sometimes called a *critical-flow* nozzle or orifice, or a *critical-flow prover*. The critical-flow prover is used both as a flow regulator and as a meter for measuring rate of discharge. Accuracies within one per cent are claimed for critical-flow meters.

111. Flow through a converging-diverging nozzle

The next step is to investigate the flow of a gas in a converging-diverging nozzle, of the type developed by the Swedish engineer de Laval. Attention will be directed to the pressure at different points along the nozzle. Assume that a certain nozzle shape is given, as shown in Fig. 140a. Let A be the cross-sectional area at any point, and x the distance along the axis of the nozzle. The weight rate of discharge W is the same for each cross-section. Thus the ratio W/A becomes a known function of x. Fig. 140b shows a plot of this function for one value of W; the W selected equals the maximum rate that can pass through the nozzle for given inlet and exit conditions.

The pressure at points along the nozzle can be determined by an adaptation of Equation (146). Let p_1 be the entrance pressure, p the pressure at any point, and V the velocity at any point in the nozzle. With this notation, Equation (146) becomes

$$V = \sqrt{\left(\frac{2k}{k-1}\right)\frac{p_1}{\rho_1}\left[1 - \left(\frac{p}{p_1}\right)^{\frac{k-1}{k}}\right]} \tag{151}$$

Since $W = AV/v$, and $pv^k = p_1v_1{}^k$, Equation (151) gives

$$\frac{W}{A} = \frac{V}{v} = \sqrt{\left(\frac{2gk}{k-1}\right)\frac{p_1}{v_1}\left[\left(\frac{p}{p_1}\right)^{\frac{2}{k}} - \left(\frac{p}{p_1}\right)^{\frac{k+1}{k}}\right]}. \tag{152}$$

Figure 140c shows a plot of W/A against p/p_1 for the given W and nozzle shape. Fig. 140b gives a certain value of W/A for each value of x;

Fig. 140c gives the corresponding value of p. Fig. 140d shows the resulting pressure variation p along the axis of the nozzle.

Point B in Fig. 140d represents the entrance pressure p_1. The fluid velocity increases in the converging tube as the pressure drops from p_1 to the critical pressure p_c (from point B to D). At the throat the fluid velocity equals the velocity of pressure propagation. Note that the

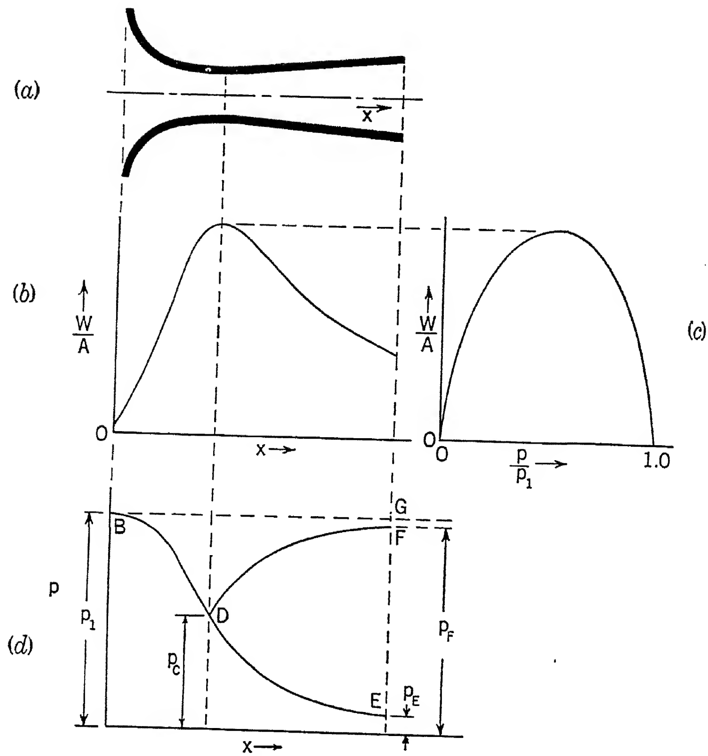

FIG. 140. Flow through a converging-diverging nozzle.

process from B to D is unique. For the diverging portion one value of x gives one value of W/A, and *two* values of the pressure p. The gas can expand adiabatically, or it can be compressed adiabatically in the diverging portion. Flow beyond point D in Fig. 140d depends upon the exit conditions; there are two and only two possibilities for frictionless flow:

(1) If the gas expands adiabatically in the diverging portion, the process follows the curve D to E in Fig. 140d. There is only one exit

pressure p_E (for the given nozzle shape) which will make this expansion possible. Figure 133 shows that during this expansion process the fluid velocity *increases above* the velocity of sound. The channel is diverging, and the local Mach's number at each point beyond the throat is greater than unity.

(2) If the gas is compressed adiabatically without friction losses, the pressure in the diverging portion follows the curve D to F in Fig. 140d. There is only one exit pressure p_F which will make this compression possible. Beyond the throat the fluid velocity *decreases* below the velocity of sound; the local Mach's number at each point beyond the throat is less than unity. The fluid velocity and the pressure at the

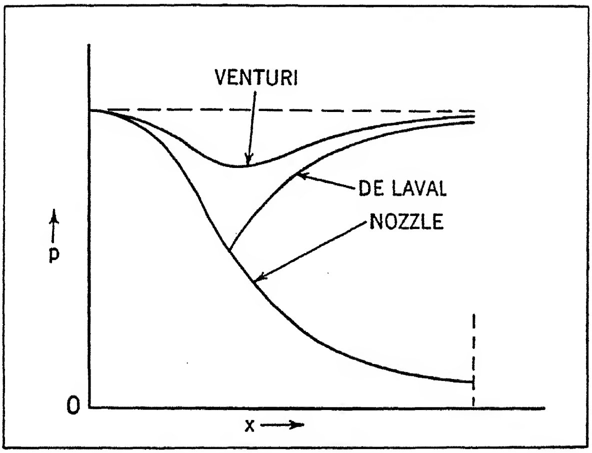

FIG. 141. Pressure distribution for a venturi tube and a de Laval nozzle.

exit are the same as those in the section of the same area in the converging portion of the nozzle.

The foregoing relations can be employed to demonstrate that the rate of discharge W is a maximum for the two pressures p_E and p_F. There is the question regarding the flow when the actual exit pressure lies between p_E and p_F. Experiments by Stodola and by Buechner show that flow with exit pressures between p_E and p_F is impossible without losses. For such exit pressures the gas, after flowing through the throat, first expands at a velocity above the velocity of sound. Then, at some point in the diverging portion of the nozzle, the gas is suddenly compressed; there is a loss in velocity and kinetic energy as the so-called compression shocks occur. Some kinetic energy is degraded into unavailable thermal energy.

There remains the question regarding flow when the exit pressure lies between p_1 and p_F. If the exit pressure equals p_1, the pressure distribution in the nozzle is simply constant, and is represented by the horizontal line BG in Fig. 140; the weight rate of discharge is zero. For exit pressures between p_F and p_1, the pressure distribution curve lies between the line BG and the line BDF. For such exit pressures, application of the type of diagrams in Figures 140b and 140c show that the weight rate of discharge is less than the maximum W used in the foregoing discussion. The actual throat pressure does not drop to the critical pressure, and the fluid velocity does not equal the velocity of sound. Such flow is typical of that found in venturi tubes used for metering purposes. A general comparison of the pressure distribution for a venturi tube and a de Laval nozzle is indicated in Fig. 141.

112. Compressible flow through fluid meters

Chapter 8 presented quantitative relations for the flow of an incompressible fluid through various common types of fluid meters. Using the

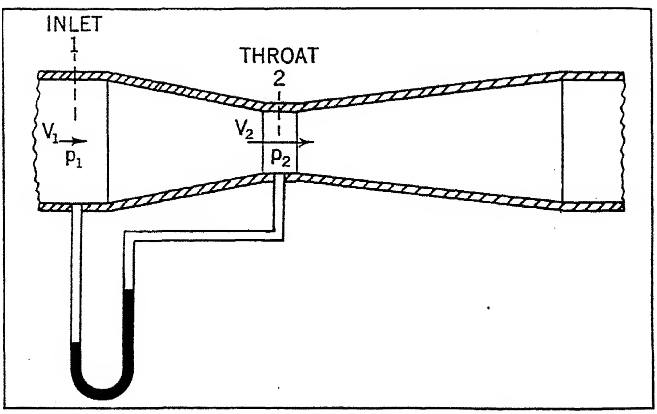

FIG. 142. Venturi meter.

notation given in Fig. 142 (same as Fig. 65), the following relation was given for the flow of an incompressible fluid through a venturi meter:

$$\text{actual } Q = CA_2V_2 = \frac{CA_2}{\sqrt{1 - \left(\frac{A_2}{A_1}\right)^2}} \sqrt{2g\left(\frac{p_1 - p_2}{w}\right)} \qquad (153)$$

where Q is the volume rate of flow and C is a dimensionless discharge coefficient. A similar relation was presented for flow nozzles and orifice meters. Equation (153) can be put into a simpler form by substituting

the flow coefficient K for the term

$$\frac{C}{\sqrt{1 - \left(\dfrac{A_2}{A_1}\right)^2}}.$$

Thus

$$Q = KA_2 \sqrt{2g\left(\frac{p_1 - p_2}{w_1}\right)}. \tag{154}$$

Let W represent the weight rate of flow (as pounds per second). Since $W = A_2V_2w = Qw$, then

$$W = KA_2w \sqrt{2g\left(\frac{p_1 - p_2}{w}\right)}. \tag{155}$$

It is common practice in fluid metering work to generalize Equation (155) by multiplying the right side by a so-called *expansion* factor, and by specifying the specific weight w as that at the inlet. Let Y represent the expansion factor. Thus, for both incompressible and compressible flow, the fundamental relation can be written as

$$W = KA_2w_1Y \sqrt{2g\left(\frac{p_1 - p_2}{w_1}\right)}. \tag{156}$$

For incompressible fluids $Y = 1$. If the fluid is frictionless, then the discharge coefficient C in the flow coefficient K is unity. For flow with friction, the discharge coefficient should be determined from suitable experiments. Tables and charts giving values of expansion factors can be found in the reference literature, such as the A.S.M.E. reports on fluid meters. It is customary to use theoretically determined values of Y for flow nozzles and venturi tubes, and to use experimentally determined values of Y for orifice meters. The minimum jet area and the pressure at various points in an orifice meter are not the same as those in a venturi tube of the same diameter operating with the same pressure drop.

Consider the determination of Y for a gas flowing through a venturi meter or a flow nozzle. An adiabatic process is assumed; this assumption is reasonable for many practical applications. Then Equation (129) applies, and can be written as

$$\frac{V_2^2 - V_1^2}{2} = \left(\frac{k}{k-1}\right)\frac{gp_1}{w_1}\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\right]. \tag{157}$$

A combination of the continuity equation $A_2V_2/v_2 = A_1V_1/v_1$ and the adiabatic relation $p_1v_1^k = p_2v_2^k$ gives

$$V_1 = V_2\frac{A_2}{A_1}\left(\frac{p_2}{p_1}\right)^{1/k}. \tag{158}$$

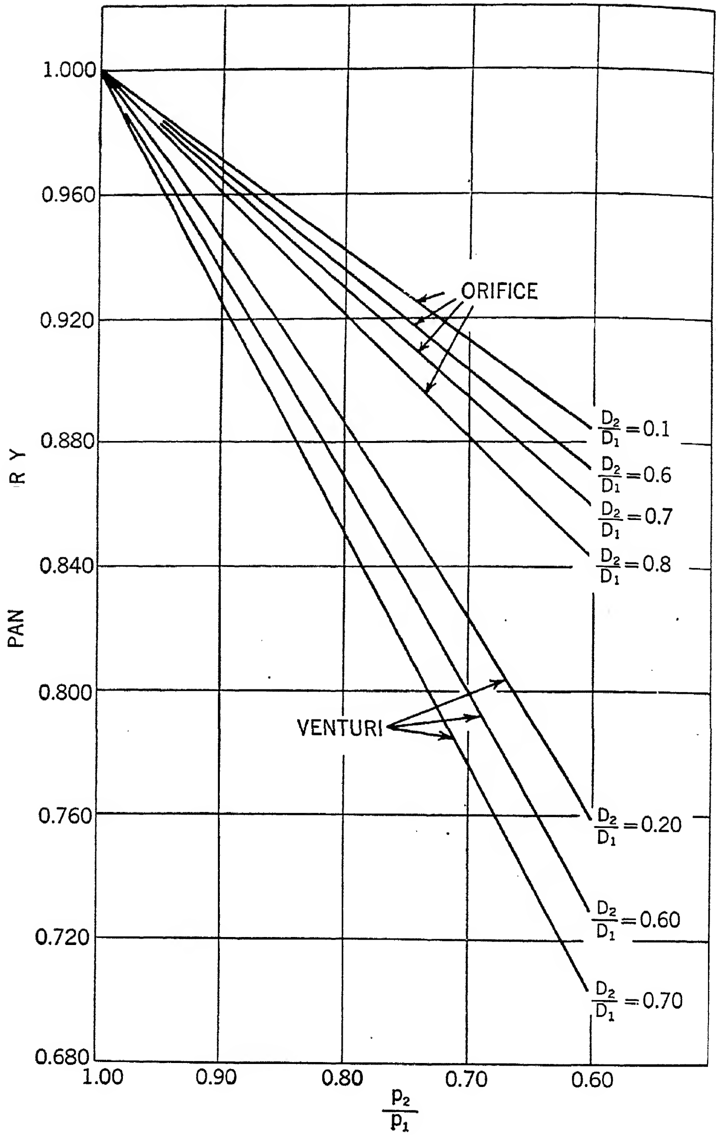

FIG. 143. Expansion ratios plotted against pressure ratio for $k = 1.4$. Orifice values are for flange, radius, and *vena contracta* taps. (Data from *Fluid Meters, Their Theory and Application.* A.S.M.E., 1937.)

V_1 can be eliminated from Equation (157) by using Equation (158). Thus

$$V_2 = \sqrt{\frac{2g\left(\frac{k}{k-1}\right)\frac{p_1}{w_1}\left[1-\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\right]}{1-\left(\frac{A_2}{A_1}\right)^2\left(\frac{p_2}{p_1}\right)^{2/k}}} \tag{159}$$

For compressible flow, then,

$$W = CA_2w_2\sqrt{\frac{\left(\frac{2gk}{k-1}\right)\frac{p_1}{w_1}\left[1-\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\right]}{1-\left(\frac{A_2}{A_1}\right)^2\left(\frac{p_2}{p_1}\right)^{2/k}}}. \tag{160}$$

Equating the W in Equation (156) to the W in Equation (160), and solving for Y, gives

$$Y = \left[\left(\frac{p_2}{p_1}\right)^{2/k}\left(\frac{k}{k-1}\right)\left\{\frac{1-\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}}{1-\left(\frac{p_2}{p_1}\right)}\right\}\left\{\frac{1-\left(\frac{A_2}{A_1}\right)^2}{1-\left(\frac{A_2}{A_1}\right)^2\left(\frac{p_2}{p_1}\right)^{2/k}}\right\}\right]^{1/2} \tag{161}$$

At first thought it might appear that the introduction of the factor Y provides a complication. Closer inspection, however, reveals the fact that Y is a function of three dimensionless ratios: the pressure ratio p_2/p_1, the area ratio A_2/A_1, and the ratio of the specific heats k. If values of Y are once determined, they will be convenient for further calculations. Some values of the expansion factor are plotted in Fig. 143. The curves marked "venturi" are also applied to flow nozzles. D_2 is the throat or orifice diameter, and D_1 is the inlet or pipe diameter.

113. Drag at high velocities

Generally speaking, at low velocities (below the velocity of sound) the drag of a body is low if the body is streamlined with a blunt nose and a gradually tapering tail. The main effect of the tail shape is to reduce the eddying wake. For bodies moving at high velocities (at or above the velocity of sound) the nose shape has a large influence on the resistance; a desirable shape for low drag is a pointed nose. The pointed nose serves to reduce the effect of the compression shock waves. Figure 144 shows the effect of nose shape on the drag coefficients of various projectiles. At high velocities the drag is primarily a function of Mach's number. Figure 144 indicates a sharp increase in drag coefficient as Mach's number approaches and exceeds unity. The lower value of C_D is due to a pointed nose and is clearly indicated.

For lifting-vane sections (such as wings and propeller blade elements), compressibility effects may be appreciable in cases where the undisturbed Mach's number is less than unity. The local velocity in some region over a surface may be above the local acoustic velocity; the local Mach's

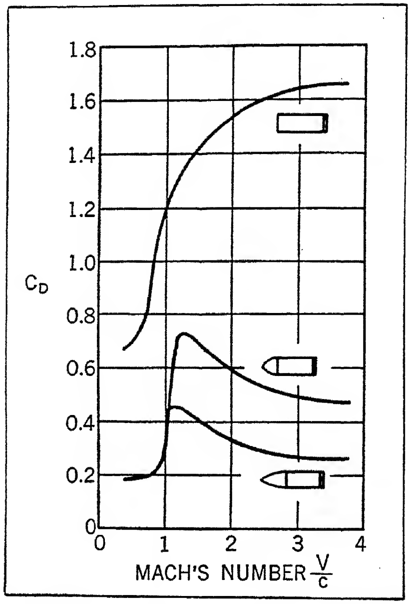

FIG. 144. Drag coefficients for a 15-centimeter projectile with various shapes of head. (Data adapted from *The Mechanical Properties of Fluids*, chap. 10, page 361, by F.R.W. Hunt. Blackie and Sons, London, 1937.)

number may be greater than unity, even though the undisturbed Mach's number is less than unity.

SELECTED REFERENCES

Flow of Gases at a Rate Exceeding the Acoustic Velocity by O. G. Tietjens, *A.S.M.E. Transactions*, APM-53-4, vol. 53, 1931, page 49.

The Physics of Solids and Fluids by P. P. Ewald, T. Pöschl, and L. Prandtl. Blackie and Son, London, 1936.

Steam and Gas Turbines by A. Stodola, vol. I. McGraw-Hill, New York, 1927.

PROBLEMS

115. How long does it take a pressure wave to travel 1 mile through water at ordinary conditions?

116. In an undisturbed stream of carbon dioxide the pressure is 15.0 pounds per square inch absolute, the density is 0.0032 slugs per cubic foot, and the velocity is 500 feet per second. If a body were held stationary in the stream, what would be the stagnation-point pressure on the basis of compressible flow? What would be the stagnation pressure if the fluid were assumed incompressible?

117. At point A in the undisturbed portion of an airstream which flows past a body, ρ is 0.002378 slugs per cubic foot, the pressure is 14.7 pounds per square inch absolute, the velocity is 450 feet per second, and c is 1120 feet per second. The pressure at a point B on the body is 8.0 pounds per square inch absolute. Calculate the Mach's number for each point.

118. A photograph of a bullet shows a Mach's angle of 28°. Estimate the speed of the bullet for standard air.

119. Air flows through a converging tube having a throat area of 0.50 square inch. The entrance pressure is 140 pounds per square inch absolute, the entrance temperature is 120° Fahrenheit, and the nozzle discharges into the atmosphere at 14.7 pounds per square inch absolute. If the entrance velocity is negligible, what is the weight discharge per unit time?

120. Starting with Equation (134), derive Equation (135).

121. Derive Equation (144) from Equation (128).

122. When the receiver pressure is equal to or less than the critical pressure, the following relation is sometimes given for the flow of air through an orifice or a short converging tube:

$$W = 0.53 \frac{Ap_1}{\sqrt{T_1}}.$$

W represents the weight rate of flow per unit time, A the throat area, p_1 the entrance pressure, and T_1 the absolute temperature at entrance. This relation is commonly called Fliegner's equation. Derive this special relation, starting with Equation (146).

CHAPTER 13

Flow of Compressible Viscous Fluids in Pipes

> Most engineers have but a hazy idea of the physical relations involved in gas flow at high velocities. It is a subject that might well be given much more attention by many of our engineering schools.—H. S. BEAN.[1]

The steady flow of incompressible fluids in pipes was discussed in Chapter 7. The relations for incompressible flow apply directly to a good many cases in which the fluid is a liquid. These relations also apply directly to many cases in which the fluid is a gas; with relatively short pipe lengths and low velocities, for example, the pressure drop, and consequently the density change, may not be large. There are other applications, like those with very long pipes, in which the pressure drop and the density change may be appreciable. Long natural-gas lines, municipal gas-supply systems, various industrial applications in which gas is transported from one point to another, and air ducts for heating and ventilating might be cited as examples. The present chapter extends the foregoing relations by taking into account elastic or compressibility effects for two practical, limiting flows: (1) isothermal, and (2) adiabatic. The following discussion is confined to the flow in a pipe of constant diameter.

114. Compressible flow in pipes at constant temperature

When a compressible fluid flows in a long pipe, the pressure gradually decreases because of frictional resistance; the specific volume increases if the temperature remains constant. The steady isothermal flow in a pipe will be analyzed by putting the energy equation in differential form, expressing the frictional resistance term as a function of differential length, and then integrating. Three different possibilities will be investigated: (a) flow in horizontal pipes; (b) upward flow in long vertical pipes; and (c) downward flow in long vertical pipes.

(a) *Flow in Horizontal Pipes.* Let dl represent the length of an infinitesimal element along a horizontal pipe of internal diameter D. The specific volume of the fluid in this element is v. The pressure drop across the element is dp, and the velocity change is dV. The general

[1] From discussion of the paper *Flow of Gases at a Rate Exceeding the Acoustic Velocity* by O. G. Tietjens, *A.S.M.E. Transactions*, APM-53-4, 1931, page 49.

energy equation, in differential form, can be written as

$$vdp + \frac{d}{2g}(V^2) + dR = 0. \tag{162}$$

where dR is the energy (in FL/F units, like foot-pounds per pound of fluid) or lost head used to overcome the frictional resistance. Following the procedure employed in Chapter 7, dR will be expressed as

$$dR = f\frac{V^2}{2gD}dl. \tag{163}$$

The general energy equation then becomes

$$vdp + \frac{VdV}{g} + \frac{fV^2}{2gD}dl = 0 \tag{164}$$

or

$$\frac{2gv}{V^2}dp + 2\frac{dV}{V} + \frac{fdl}{D} = 0. \tag{165}$$

Several terms in Equation (165) require investigation before integrating. Note that D is constant. Experimental results show that f is a function of Reynolds' number $N_R = \rho VD/\mu$. For engineering purposes it is common to regard the dynamic viscosity as a function of temperature only; for an isothermal process the dynamic viscosity is constant. Let W represent the weight rate of flow. Then $W = AV/v$, where A is the constant pipe area; V and v each vary along the pipe. Reynolds' number becomes

$$N_R = \frac{\rho VD}{\mu} = \frac{DW}{\mu gA}. \tag{166}$$

For a given weight rate of discharge, N_R is constant along the pipe; hence the friction factor is constant along the pipe. The friction factor can be determined from Fig. 49.

The continuity equation and the equation of state can be combined to put the first term of Equation (165) into a form convenient for integrating. The continuity equation states that

$$W = \frac{AV_1}{v_1} = \frac{AV}{v} = \frac{AV_2}{v_2}.$$

For gases at constant temperature $p_1v_1 = pv = p_2v_2 = RT$. Thus

$$\frac{V}{V_1} = \frac{v}{v_1} = \frac{p_1}{p}, \qquad \frac{v}{V^2} = \frac{v_1p}{V_1^2p_1}$$

Equation (165) can now be integrated with the proper limits:

$$\frac{2gv_1}{V_1^2p_1}\int_2^1 pdp + 2\int_2^1 \frac{dV}{V} + \frac{f}{D}\int_l^0 dl = 0,$$

$$p_1^2 - p_2^2 = \frac{V_1^2p_1}{gv_1}\left[2\log_e\frac{V_2}{V_1} + \frac{fl}{D}\right]. \tag{167}$$

If the pipe is long, the term $2 \log_e V_2/V_1$ is negligible in comparison with the term fl/D; that is, the kinetic energy change is negligible, and Equation (167) becomes

$$\frac{p_1^2 - p_2^2}{p_1^2} = \frac{flV_1^2}{gDp_1v_1} \tag{168}$$

$$p_2 = p_1\sqrt{1 - \frac{flV_1^2}{Dgp_1v_1}}. \tag{169}$$

The pressure drop becomes

$$p_1 - p_2 = p_1\left[1 - \sqrt{1 - \frac{flV_1^2}{gDp_1v_1}}\right]. \tag{170}$$

Expansion of the right side of Equation (170) in terms of a power series shows that if the term flV_1^2/gDp_1v_1 is very small in comparison with unity, then the pressure drop is

$$p_1 - p_2 = \frac{flV_1^2}{2gDv_1}. \tag{171}$$

which is the relation for incompressible flow.

(b) *Upward Flow in Long Vertical Pipes.* For upward flow in a vertical pipe there is an additional pressure decrease due to the vertical rise. Thus

$$dp = \frac{fV^2dl}{2gDv} + \frac{dl}{v}. \tag{172}$$

Integrating Equation (172) and substituting limits yields, as a final result,

$$l = \frac{p_1v_1}{2}\log_e\left[\frac{p_1^2 + \dfrac{fV_1^2p_1^2}{2gD}}{p_2^2 + \dfrac{fV_1^2p_1^2}{2gD}}\right]. \tag{173}$$

(c) *Downward Flow in Long Vertical Pipes.* For downward flow in a vertical pipe there is a reduction in the pressure drop due to the vertical decrease in height. Thus

$$dp = \frac{fV^2dl}{2gDv} - \frac{dl}{v}. \tag{174}$$

Integrating Equation (174) and substituting limits yields, as a final result,

$$l = \frac{p_1 v_1}{2} \log_e \left[\frac{p_2^2 - \dfrac{f V_1^2 p_1^2}{2gD}}{p_1^2 - \dfrac{f V_1^2 p_1^2}{2gD}} \right]. \tag{175}$$

115. Comparison of incompressible flow and compressible isothermal flow relations

The relations for the flow of a compressible fluid in a pipe are more complicated than the simple relation for the flow of an incompressible fluid. The question is frequently raised as to the limits of application of the incompressible-flow relation. It is advisable to check with the more general relations for compressible flow if there is any question of application. The question of application frequently depends upon the accuracy desired. Sometimes the rule is given that the incompressible flow relation can be applied to isothermal compressible flow problems if the pressure drop $(p_1 - p_2)$ is less than 10 per cent of the initial pressure p_1. A comparison will be drawn between the incompressible-flow relation and the isothermal compressible-flow relation for long horizontal pipes. This comparison will be helpful in answering questions regarding application.

It will be convenient to express the pertinent relations in terms of dimensionless ratios. Equations (170) and (171) then become:

(compressible flow) $$\frac{p_1 - p_2}{p_1} = 1 - \sqrt{1 - \frac{flV_1^2}{gDp_1v_1}},$$

(incompressible flow) $$\frac{p_1 - p_2}{p_1} = \frac{flV_1^2}{2gDp_1v_1}.$$

Let the dimensionless ratio $flV_1^2/2gDp_1v_1$ be denoted by B. The ratio B can be determined from initial conditions in the pipe. Then the foregoing relations can be expressed in the simple forms

(compressible flow) $$\frac{p_1 - p_2}{p_1} = 1 - \sqrt{1 - 2B}, \tag{176}$$

(incompressible flow) $$\frac{p_1 - p_2}{p_1} = B. \tag{177}$$

Equations (176) and (177) can be compared for identical values of B. Figure 145 shows a plot of the pressure-drop ratio for compressible flow against the pressure-drop ratio for incompressible flow.

There is not much difference between the incompressible and compressible relations for values of $\dfrac{p_1 - p_2}{p_1}$ less than 0.1. The difference

becomes greater, however, at values of the pressure-drop ratio greater than 0.1. For example, for an incompressible value of $\frac{p_1 - p_2}{p_1} = 0.30$,

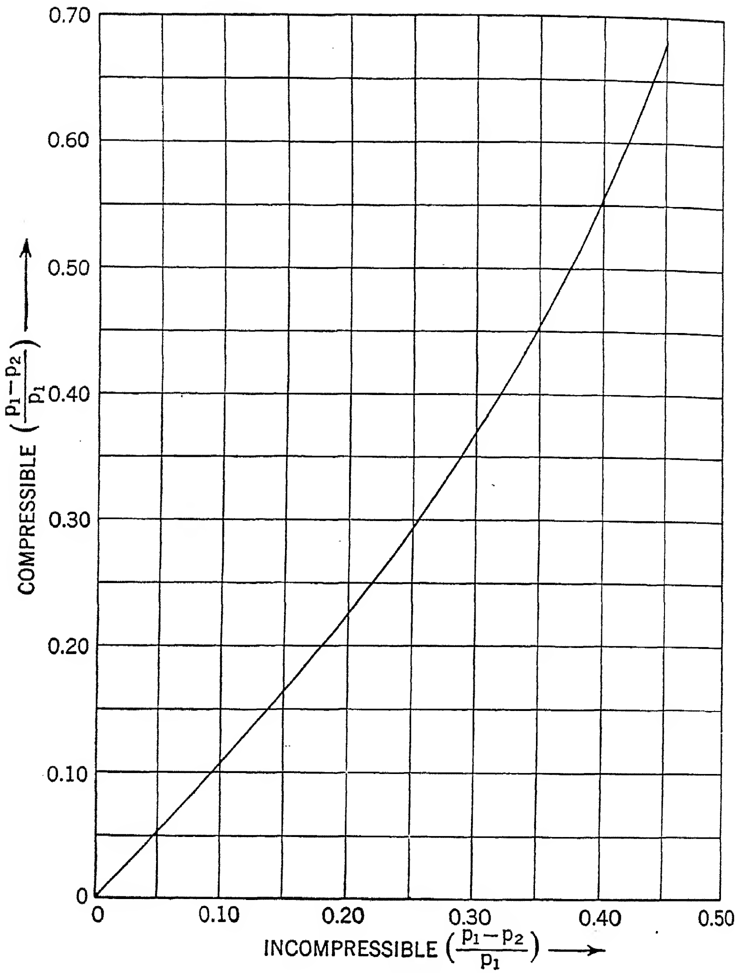

FIG. 145. Pressure-drop ratio for isothermal compressible flow plotted against pressure-drop ratio for incompressible flow in long horizontal pipes.

the corresponding compressible value of $\frac{p_1 - p_2}{p_1}$ is nearly 0.37. The pressure drop for isothermal compressible flow can be determined by

first using the simple relation for incompressible flow, and then referring to Equation (176) or Fig. 145.

116. Flow of gases in insulated pipes

The present article and the next two will be devoted to the steady flow of a gas in a horizontal pipe so insulated that no heat passes through the pipe walls. If the flow were frictionless, then the process would follow the simple reversible adiabatic relation pv^k = constant. If there is friction, however, the process is irreversible, and does not follow this simple relation. The relation for the actual process will be derived by combining the energy equation and the equation of continuity.

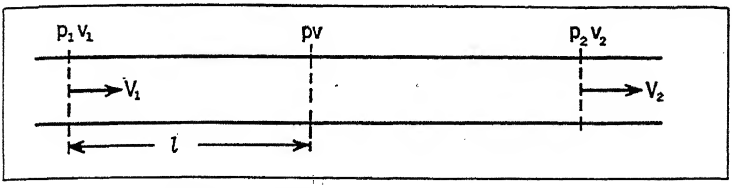

FIG. 146. Notation for flow in horizontal pipe.

Equation (128) gives the energy equation for the adiabatic flow of a gas in a horizontal pipe. Using the notation shown in Fig. 146, the energy equation states that

$$\frac{V_2^2}{2g}\left(\frac{k-1}{k}\right) + p_2v_2 = \frac{V_1^2}{2g}\left(\frac{k-1}{k}\right) + p_1v_1$$

or

$$pv\left[1 + \frac{(k-1)V^2}{k2gpv}\right] = \text{constant.} \tag{178}$$

Equation (178) gives the actual equation of state for a gas flowing in a pipe with friction. The term $kgpv$ equals $kp/\rho = c^2$, where c is the velocity of pressure propagation corresponding to the pressure p. Equation (178) can then be expressed in an alternate form using Mach's number N_M.

$$pv\left[1 + \left(\frac{k-1}{2}\right)N_M{}^2\right] = \text{constant.} \tag{179}$$

If Mach's number is very small in comparison with unity, then the relation between pressure and specific volume is approximately isothermal; that is, pv = constant. In many applications, however, N_M is not negligible, and the pressure-volume relation is correctly expressed by either Equation (178) or (179). If W is the constant rate of flow, and

A is the constant pipe area, then $W = AV/v$, and

$$pv + \left(\frac{k-1}{k}\right)\frac{W^2 v^2}{2gA^2} = \text{constant.} \tag{180}$$

Note that the factor $\left(\frac{k-1}{k}\right)\frac{W^2}{2gA^2}$ is a constant for a particular problem of steady flow.

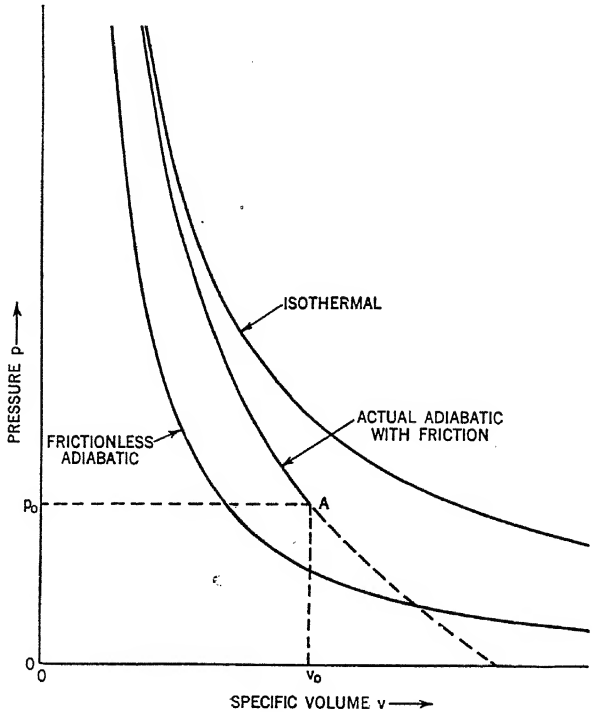

FIG. 147. Comparison of different gas processes for the same initial pressure and specific volume.

Figure 147 provides a graphical comparison of the different pressure-volume processes. The isothermal process follows the simple relation pv = constant, and the frictionless adiabatic follows the relation pv^k = constant. The actual curve, the adiabatic process with friction, was drawn according to Equation (180) for a certain value of the ratio W/A. Each curve was drawn for the same initial pressure and specific volume. Actual flow is *not* realized below a certain limiting or critical pressure p_0 (at point A). The dotted portion of the actual curve in Fig. 147 has

no physical significance—for one thing, it approaches the physically impossible condition of finite volume at zero pressure. Limiting pipe flow is similar in some respects to limiting nozzle flow. The dotted curve in Fig. 147 is not realized for reasons similar to those which show that the dotted portion of the curve for nozzle flow in Fig. 139 is not realized.

Stodola, Schüle, and other investigators[2] have established the following results for the flow of a gas in an insulated pipe of constant diameter:

> (a) It is impossible for an expanding flow in a cylindrical pipe to exceed the acoustic velocity.
>
> (b) The limiting pressure p_0 and the limiting specific volume v_0 are reached when the gas velocity equals the acoustic velocity.
>
> (c) This limiting, acoustic velocity is reached *only* at the pipe end (for a certain length). A further decrease in the back pressure at the exit of the pipe would not influence the pipe flow in any way, for the gas is moving at the velocity of pressure propagation, and the lower back pressure cannot be telegraphed back into the pipe.

The limiting condition of adiabatic flow with friction is reached when the friction is reduced to zero. The change of state without friction is expressed by the relation pv^k = constant. The actual curve loses its physical significance when its slope equals that of the frictionless adiabatic.[3] The slope of the reversible adiabatic curve pv^k = constant is

$$\frac{dp}{dv} = -k\frac{p}{v}. \tag{181}$$

The slope of the actual curve can be found by differentiating Equation (180):

$$\frac{dp}{dv} = -\left[\left(\frac{k-1}{k}\right)\frac{W^2}{gA^2} + \frac{p}{v}\right. \tag{182}$$

Equating the slope given by Equation (181) to that given by Equation (182) for the limiting values of p_0, v_0, and V_0 gives the result $V_0 = \sqrt{kgp_0v_0} = c_0$. The fluid velocity at p_0 and v_0 thus equals the velocity of pressure propagation at p_0 and v_0. Article 117 discusses the determination of the limiting pressure, specific volume, and velocity without regard to the length of pipe. Article 118 discusses the conditions along the pipe length.

[2] See references at end of this chapter.

[3] An elegant demonstration of this limiting condition can be given by an entropy analysis. See *Steam and Gas Turbines* by A. Stodola. McGraw-Hill, New York, 1927, vol. I.

117. Limiting pressures and velocities for adiabatic flow

Consider the flow of a gas along a horizontal pipe for a certain initial pressure p_1, an initial specific volume v_1, and an initial Mach's number N_{M_1}. At the limiting pressure p_0 the limiting Mach's number $N_{M_0} = V_0/c_0$ is unity. From Equation (179)

$$p_0v_0\left[1 + \frac{k-1}{2}\right] = p_1v_1\left[1 + \left(\frac{k-1}{2}\right)N_{M_1}{}^2\right]. \tag{183}$$

From the equation of continuity

$$\frac{W}{A} = \frac{V}{v} = \frac{V_1}{v_1} = \frac{V_0}{v_0}. \tag{184}$$

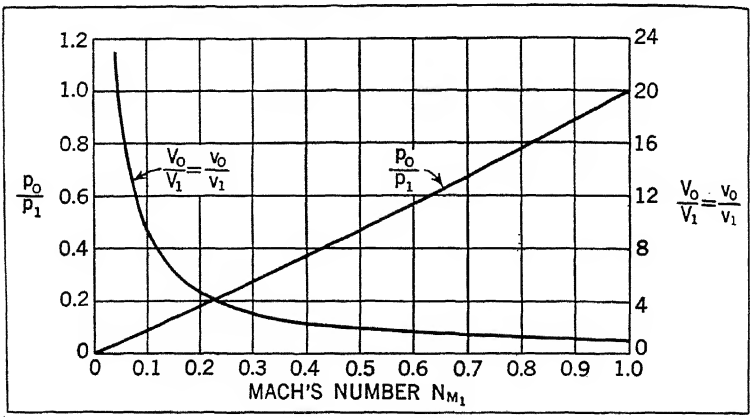

FIG. 148. Limiting values of velocity, pressure, and specific volume ratios for flow in an insulated pipe for gases with $k = 1.4$.

The acoustic velocity at the initial state is $c_1 = \sqrt{kgp_1v_1}$, and the acoustic velocity at the limiting condition is $c_0 = V_0 = \sqrt{kgp_0v_0}$. Combining these two relations with Equation (183) gives

$$\frac{p_0v_0}{p_1v_1} = \frac{c_0{}^2}{c_1{}^2} = \left(\frac{k-1}{k+1}\right)\left[\frac{2}{k-1} + N_{M_1}{}^2\right],$$

$$\frac{c_0}{V_1} = \left\{\left(\frac{k-1}{k+1}\right)\left[1 + \frac{2}{N_{M_1}{}^2(k-1)}\right]\right\}^{1/2}. \tag{185}$$

Using the relations expressed in Equation (184) gives

$$\frac{v_0}{v_1} = \left\{\left(\frac{k-1}{k+1}\right)\left[1 + \frac{2}{(k-1)N_{M_1}{}^2}\right]\right\}^{1/2}. \tag{186}$$

Substituting the foregoing relation in Equation (183) yields, finally,

$$\frac{p_0}{p_1} = N_{M_1}{}^2 \left\{ \left(\frac{k-1}{k+1} \right) \left[1 + \frac{2}{(k-1)N_{M_1}{}^2} \right] \right\}^{\frac{1}{2}}. \tag{187}$$

Equations (186) and (187) provide means for calculating the limiting pressure and specific volume. These equations show that each ratio is dependent only on k, the ratio of specific heats, and the initial Mach's number N_{M_1}. The limiting velocity c_0, which equals V_0, can be determined by Equation (185). Figure 148 shows values of the foregoing ratios plotted against N_{M_1} for a value of $k = 1.40$. Table 2, page 5, shows that $k = 1.40$ for air, carbon monoxide, hydrogen, and oxygen.

118. Conditions along the length of an insulated pipe

The variation in pressure, specific volume, and velocity along the pipe will be investigated by putting the fundamental relations in differential form, expressing the frictional resistance as a function of differential length, and then integrating. The functional relation between the variables velocity and length will be obtained first. Various methods could be employed for getting this relation; the following discussion details one method.

Let dl represent the length of an infinitesimal element along a horizontal insulated pipe. The specific volume of the fluid in this element is v. The pressure drop across the element is dp, and the velocity change is dV. Equation (164) shows that the general energy equation for these conditions can be put in the form

$$vdp + \frac{VdV}{g} + \frac{fV^2}{2gD}\,dl = 0. \tag{188}$$

Equation (188) can be written as

$$d(pv) - pdv + \frac{VdV}{g} + \frac{fV^2}{2gD}\,dl = 0. \tag{189}$$

Equation (178) can be expressed as

$$pv + \left(\frac{k-1}{k} \right) \frac{V^2}{2g} = G \tag{190}$$

where G is a constant. Differentiation of Equation (190) yields

$$d(pv) = -\left(\frac{k-1}{k} \right) \frac{VdV}{g}. \tag{191}$$

Differentiation of the equation of continuity, $W = AV/v =$ constant. gives

$$Wdv = AdV \qquad dv = \frac{vdV}{V}. \tag{192}$$

Combining the relation $pdv = pv(dV/V)$ with Equation (190) gives

$$pdv = \left[G - \left(\frac{k-1}{k}\right)\frac{V^2}{2g}\right]\frac{dV}{V}. \tag{193}$$

Substitution of Equations (191) and (193) in Equation (189) furnishes the final differential relation

$$\frac{fdl}{2D} = gG\frac{dV}{V^3} - \frac{dV}{V}\left(\frac{k+1}{2k}\right). \tag{194}$$

Several terms in Equation (194) require investigation before an integration can be made. D and G are each constant. f is a function of Reynolds' number. Equation (166) shows that N_R is a function of the dynamic viscosity alone, for the particular conditions taken. The viscosity, in turn, is primarily a function of temperature. The change in viscosity and f is usually small, particularly at high Reynolds' numbers. f will be regarded as constant for integration purposes. In a practical problem, the initial and final Reynolds' numbers can be computed and an average value of f used.

Direct integration of Equation (194) between points 0 and l, at which the velocities are V_1 and V respectively, determines the dimensionless ratio

$$\frac{fl}{D} = -gG\left[\frac{1}{V^2} - \frac{1}{V_1^2}\right] - \left(\frac{k+1}{k}\right)\log_e\frac{V}{V_1}. \tag{195}$$

Inserting the evaluation of G for the initial point in Equation (195) gives the final form:

$$\frac{fl}{D} = -\left(\frac{k+1}{k}\right)\log_e\frac{V}{V_1} + \frac{1}{k}\left[\frac{1}{N_{M_1}^2} + \frac{k-1}{2}\right]\left[1 - \frac{V_1^2}{V^2}\right] \tag{196}$$

The ratio v/v_1 can be substituted for its equivalent V/V_1 in Equation (196). Equation (186) gives the limiting value of the ratio v_0/v_1 as a function of k and the initial Mach's number N_{M_1}. The maximum length of pipe for given initial conditions (given p_1, v_1, and N_{M_1}) can be determined by substituting the limiting ratio v_0/v_1 in Equation (196). At the end of this maximum length the fluid velocity equals the velocity of sound at p_0 and v_0. The use of a pipe longer than this maximum length would result in failure to realize the given initial conditions. For a certain pipe, an initial V_1 (or rate of flow W/A) is only possible for pipe lengths up to a definite limiting value of l. A reduction of the back pressure at the end of this maximum length below p_0 would not affect the flow *in* the pipe.

The pressure p at any point along the pipe can be expressed, by employing Equation (178), as

$$\frac{p}{p_1} = \frac{V_1}{V}\left[1 + \frac{(k-1)N_{M_1}{}^2}{2}\left(1 - \frac{V^2}{V_1{}^2}\right)\right]. \tag{197}$$

The complete solution to the problem of gas flow through an insulated pipe is now contained in Equations (186), (187), (196), and (197). A

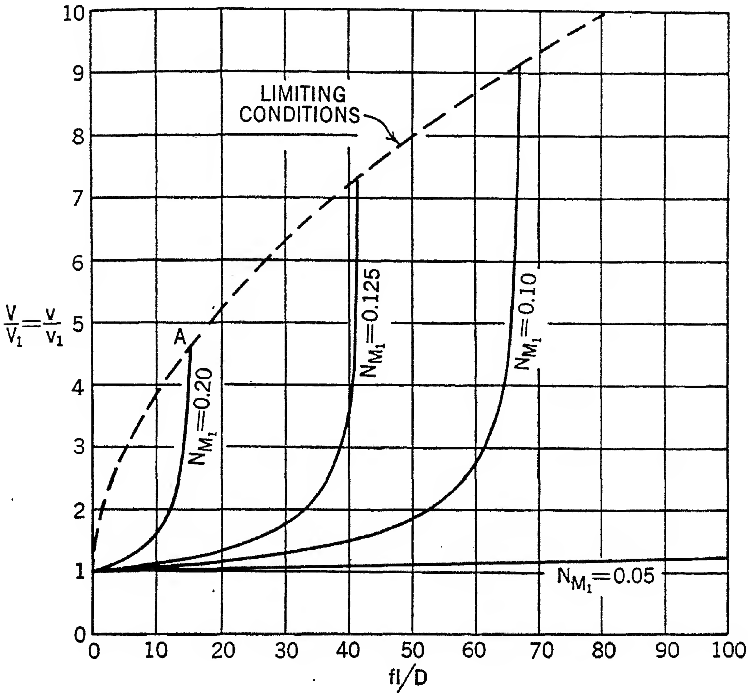

FIG. 149. Variation of velocity and specific volume ratios with Mach's number and fl/D for gas flow in an insulated pipe—for gases with $k = 1.4$.

convenient method for using and interpreting these equations is one in which the various dimensionless ratios are represented graphically for different initial conditions.

Figure 149 shows a plot of velocity and specific volume ratios versus fl/D for different initial Mach's numbers. The dotted curve in Fig. 149 represents limiting conditions. For example, for an initial $N_{M_1} = 0.20$, the point of intersection A gives the maximum value of the fl/D ratio, and the limiting values $V_0/V_1 = v_0/v_1$.

Figure 150 shows a plot of the pressure ratio against fl/D for different initial Mach's numbers. The dotted curve represents limiting conditions. For example, for an initial $N_{M_1} = 0.20$, the point of intersection A gives the maximum value of the fl/D ratio and the corresponding limiting pressure ratio p_0/p_1. Values of pressure between p_1 and p_0 at corresponding lengths can be found from Fig. 150. For example, for

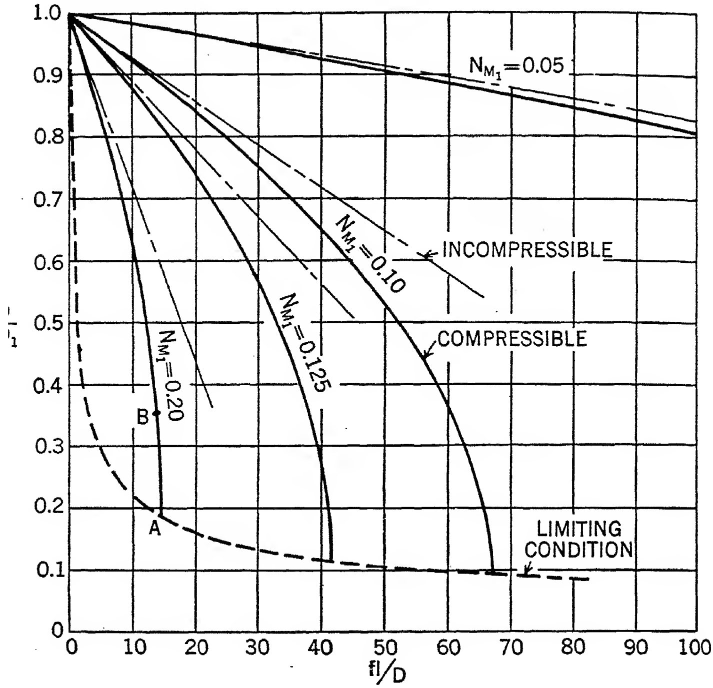

FIG. 150. Variation of pressure ratio with Mach's number and fl/D for gas flow in an insulated pipe—for gases with $k = 1.4$.

$N_{M_1} = 0.20$, the point of intersection B gives the pressure ratio p/p_1 for a certain corresponding fl/D.

119. Comparison of incompressible and compressible adiabatic relations

The pressure drop for the flow of an incompressible fluid in a horizontal pipe can be written as

$$p_1 - p = \frac{flV_1^2}{2gDv_1} \qquad \text{or} \qquad \frac{p}{p_1} = 1 - \frac{flV_1^2}{2gDp_1v_1}.$$

Since $c_1 = \sqrt{kgp_1v_1}$, the foregoing pressure ratio can be expressed in the form

$$\frac{p}{p_1} = 1 - \frac{k}{2}\left(N_{M_1}{}^2\right)\frac{fl}{D}. \tag{198}$$

Equation (198) was employed to plot the straight dot-dash lines in Fig. 150. A comparison of a dot-dash line and a solid curve, for the same N_{M_1}, brings out the differences between compressible flow and incompressible flow. At very low Mach's numbers the difference between the incompressible and the compressible-flow relations is not very great. The difference becomes greater, however, as Mach's number is increased. At small lengths (or small values of fl/D), there is not much difference between the incompressible and the compressible flow relations. The difference, however, becomes greater as the length is increased. As a general recommendation, in working problems involving the flow of gases, it is good practice to take compressibility into account unless definite evidence is available to show that the assumption of an incompressible fluid does not involve any serious errors.

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Steam and Gas Turbines by A. Stodola. McGraw-Hill, New York, 1937, vols. I and II.

Principles of Thermodynamics by G. A. Goodenough. Holt, New York, 1927.

PROBLEMS

123. Derive Equation (173) from Equation (172).

124. Derive Equation (175) from Equation (174).

125. Gas at 60° Fahrenheit enters a smooth horizontal pipe 25 miles long, of 17.5-inch diameter. The inlet pressure is 350 pounds per square inch absolute, and the inlet velocity is 10 feet per second. The gas constant R is 95.0 feet per degree Fahrenheit, and the dynamic viscosity is 110.0×10^{-6} poise. What is the pressure at the end of the pipe for isothermal flow?

126. Air enters a clean galvanized horizontal pipe of 6-inch diameter at 85 pounds per square inch absolute, 68° Fahrenheit, and with a velocity of 110 feet per second. The length is 700 feet. Find the pressure drop for: (a) incompressible flow, and (b) compressible isothermal flow.

127. 924 pounds of air per minute are to be transported from station A to B through a smooth horizontal pipe of 8-inch diameter. At A the pressure is 90 pounds per square inch absolute, and the temperature is 68° Fahrenheit, whereas the pressure at B cannot be below 84 pounds per square inch absolute. What is the maximum length of pipe that can be used between stations A and B for isothermal flow?

128. Methane flows through a clean steel insulated pipe of 4-inch diameter at a rate of 7.1 pounds per second. At inlet the temperature is 90° Fahrenheit, and the pressure is 100 pounds per square inch absolute. What is the lowest pressure

that can be obtained in the pipe if the dynamic viscosity is 2.36×10^{-7} slug per foot-second? What is the maximum length of pipe?

129. It is proposed to pump air through a clean galvanized insulated pipe 6 inches in diameter and 3500 feet long connecting station A with station B. The inlet, at station A, is to have a pressure of 120 pounds per square inch absolute, a temperature of 140° Fahrenheit, and a velocity of 120 feet per second. Consider the pipe friction constant along the line and equal to that at entrance. If a booster pump is to be inserted in the line, the inlet pressure to the pump cannot be less than 15 pounds per square inch absolute. Is a booster pump necessary in the line? If so, what is the maximum distance from station A that the booster can be placed? What is the rate of the discharge of the line?

CHAPTER 14

Flow of Liquids in Open Channels

120. General remarks

An open channel is defined as one in which the liquid stream is not completely enclosed by solid boundaries, and thus has a free surface subjected only to atmospheric pressure. The flow in an open channel depends upon the slope of the channel bottom and the slope of the liquid surface. Natural streams, rivers, artificial canals, sewers, tunnels, and pipes not completely filled with liquids are examples of open channels. Accurate solution of problems of open-channel flow is much more difficult than solution of those of pipe flow. One distinguishing feature of the flow of liquids in open channels, as contrasted with pipe flow, is that the cross-sectional area is free to change instead of being fixed. The presence of a free surface introduces complexities of a primary magnitude. It is difficult to obtain reliable experimental data on the flow in open channels. A relatively smaller range of conditions is found in pipe flow. Most pipes are round, but open-channel sections vary from circular to the irregular forms of natural rivers; channel surface conditions range from the smoothness of timber to the roughness and irregularity in the beds of some rivers. Deciding on friction factors for open channels is more troublesome than the same task for pipe flow.

The treatment of open-channel flow is therefore somewhat more empirical than that of pipe flow. The empirical treatment, however, is the best available at present, and can yield results of practical value if cautiously applied. Water is the usual liquid involved in the flow with a free surface. The empirical relations and coefficients presented in this chapter are based upon experiments with that liquid. These relations apply only to turbulent flow, which is the more common type of flow.

121. Definitions

Sometimes the word *stage* is used in place of the word *depth*. Either word refers to the vertical height of liquid in a channel. *Steady flow* refers to a condition in which the flow characteristics at any *point*, such as velocity and pressure, do not change with time. The flow of a liquid in an open channel is said to be *uniform* when the depth, velocity, liquid cross-sectional area, slope, and other such flow elements remain *constant*

from section to section. Steady flow should not be confused with uniform flow. As indicated in Fig. 151, in uniform flow the liquid surface line is parallel to the channel bottom line. The surface slope, $S = \sin \alpha$, equals the bottom slope, $S_0 = \sin \alpha_0$.

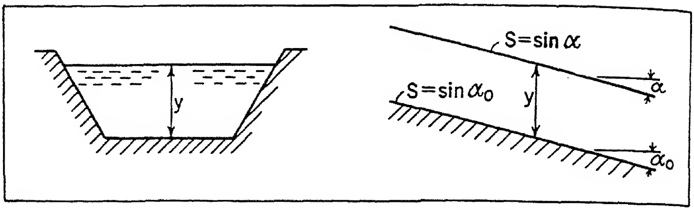

Fig. 151. Uniform flow in an open channel.

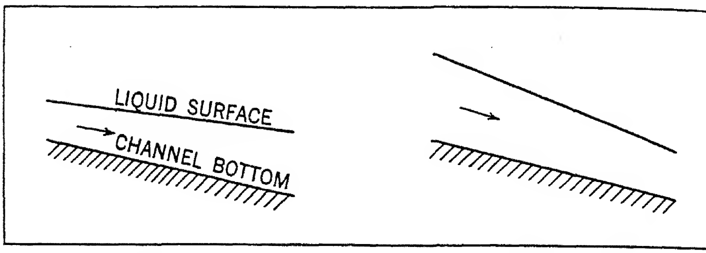

Fig. 152. Nonuniform or varied flow in an open channel.

The flow is said to be *nonuniform* or *varied* whenever the depth and other features of flow, such as the liquid cross-sectional area, the velocity, and the slope, vary from section to section. Two examples of nonuniform flow are given in Fig. 152, in which the water surface is not parallel to the bottom surface. The flow in an open channel may be uniform in one length, and nonuniform in another length.

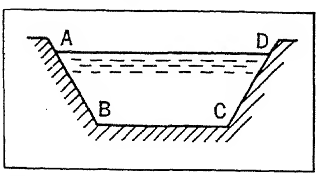

Fig. 153. Cross section of an open channel.

The term *hydraulic mean depth* or *hydraulic radius* is customarily used in open-channel studies. This term is defined as

$$\text{hydraulic mean depth} = R = \frac{\text{cross-sectional area}}{\text{wetted perimeter}}.$$

The cross-sectional area in Fig. 153 is the area $ABCD$. The wetted perimeter is the length of solid boundary in contact with the liquid, that is, the distance $ABCD$. The wetted perimeter does *not* include

the distance across the free surface. For a circular pipe flowing full, the hydraulic mean depth equals $D/4$, where D is the pipe diameter.

The following treatment applies only to channels with small slopes. Strictly speaking, the slope S should be taken as the sine of the angle of inclination. The following relations apply only to small slopes for which the sine and the tangent are very nearly equal. The diagrams in this chapter exaggerate the slope for the sake of illustration.

122. Energy relation for steady, uniform flow

The case will be taken in which the velocity V is the same at all depths. In the notation shown in Fig. 154, the general energy equation

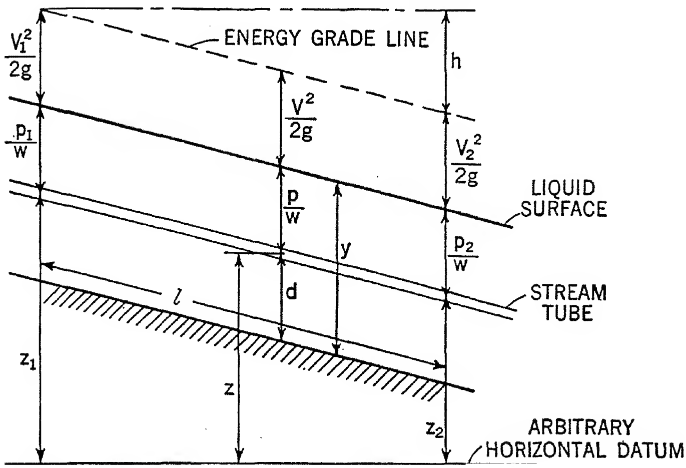

Fig. 154. Profile of channel.

between any two points 1 and 2, for the steady flow in any stream-tube, becomes

$$z_1 + \frac{p_1}{w} + \frac{V_1^2}{2g} = z_2 + \frac{p_2}{w} + \frac{V_2^2}{2g} + h, \tag{199}$$

where h is the lost head. All the streamlines are parallel to the bottom of the channel for uniform flow. Thus $p_1 = p_2$, and $V_1 = V_2$. Then

$$\text{lost head} = h = z_1 - z_2.$$

The energy required to maintain the flow in the channel is obtained at the expense of potential energy. In uniform flow the entire change in elevation is charged to the maintenance of the flow. The slope S of the energy grade line is h/l. The slope of the energy grade line equals the slope of the water surface equals the slope of the bottom only for the special case of steady, uniform flow.

123. Friction relation for steady, uniform flow

As was pointed out in Chapter 7, it is customary to express the head loss for flow in a circular pipe completely filled as

$$h = f\frac{l}{D}\frac{V^2}{2g}. \tag{200}$$

The hydraulic mean depth R for a circular pipe is $D/4$. Replacing D, substituting $S = h/l$, and solving for V, gives

$$V = \sqrt{\frac{8g}{f}}\sqrt{RS}. \tag{201}$$

For open-channel flow, it is common practice to express Equation (201) in the form

$$V = C\sqrt{RS}, \tag{202}$$

where C is a dimensional coefficient, with the dimensions $L^{1/2}T^{-1}$. Equation (202) is commonly known as the Chezy equation.

The proper evaluation of the friction coefficient C presents an important and difficult problem. Chapter 7 (Fig. 49) shows that the pipe friction coefficient f is a function of Reynolds' number and the pipe roughness. Since $C = \sqrt{8g/f}$, it follows that C is a function of the same factors, that is, C is a function of the average velocity, hydraulic mean depth, kinematic viscosity, and wall roughness. Because both applications and experiments are generally concerned with water at ordinary atmospheric temperatures, the viscosity variation is small; the viscosity has not been explicitly included in the friction relations. Apparently the available data on friction for open-channel flow are based on experiments with fairly well developed turbulent flow in rough conduits. The application of such data to flow with small depths and low velocities is questionable.

Most open channels are relatively large, as compared with pipes, and have surfaces which are rougher than pipes. Figure 49 shows that the effect of roughness becomes prominent in pipe flow at high Reynolds' numbers. Since C is a function of f, it appears that, for usual applications, the coefficient C depends largely on the character of the surface and the cross section of the channel.

A wide variety of empirical formulas for C have been presented, among them being those by Kutter, Ganguillet, Manning, and Bazin. For a general introductory study, as is given in this chapter, the Manning equation is preferable. Manning's equation states that

$$C = \frac{1.49}{n}R^{1/6}, \tag{203}$$

where n is a roughness factor. Some average values of n are listed in Table 8, with the foot taken as the length unit.

TABLE 8

SOME AVERAGE VALUES OF n FOR USE IN MANNING'S EQUATION

Nature of surface	n
Planed wood	0.012
Unplaned wood	0.013
Finished concrete	0.012
Unfinished concrete	0.014
Brick	0.016
Rubble	0.025
Earth, good condition	0.025
Earth, with stones or weeds	0.035
Gravel	0.028
Vitrified sewer pipe	0.013
Cast iron	0.014
Riveted steel	0.015

EXAMPLE. A rectangular channel lined with finished concrete is 10 feet wide. The bottom slope is 0.002, and the depth is 7 feet. Compute the rate of discharge by the Manning equation for steady, uniform flow.

$$\text{hydraulic mean depth} = R = \tfrac{70}{24}\text{ feet}$$

$$V = C\sqrt{RS} = \frac{1.49}{n}R^{1/6}\sqrt{RS} = \frac{1.49}{n}R^{2/3}S^{1/2}$$

$$Q = AV = \frac{70(1.49)}{0.012}\left(\frac{70}{24}\right)^{2/3}(0.002)^{1/2}.$$

The rate of discharge $Q = 79.4$ cubic feet per second.

124. Velocity distribution in open channels

The velocity distribution in an open channel is influenced by the walls (in the same manner as in pipes) and also by the free surface. The distribution of velocity is quite irregular in channels of variable section, particularly in natural streams. The velocity is usually highest at the point or points least affected by the solid boundaries and the free surface. Figure 155 shows some representative velocity profiles in a vertical plane. Experiments have shown that the maximum velocity in a straight section of an open channel may occur at a point below the water surface. This maximum velocity frequently does not occur at the center. No completely rational explanation of these observations has been established at the present time. Surface tension and wind effects alone cannot account for the relatively pronounced action retarding the water layers above the level of maximum velocity.

Some measurements indicate that the depth of the maximum velocity increases with the depth of the stream. The thread of maximum velocity lies very close to the surface in some shallow streams having rough beds. In some channels the vertical velocity distribution approximates a parabola whose axis is horizontal and passes through the point of maxi-

mum velocity. The velocity distribution can be represented by velocity contours, or lines of equal velocity, as shown in Fig. 156. The velocity within the area enclosed by a contour curve is higher than that at a point on the curve. The velocity outside the enclosed area is less than that on the curve. Figure 156 shows that the velocity varies from top

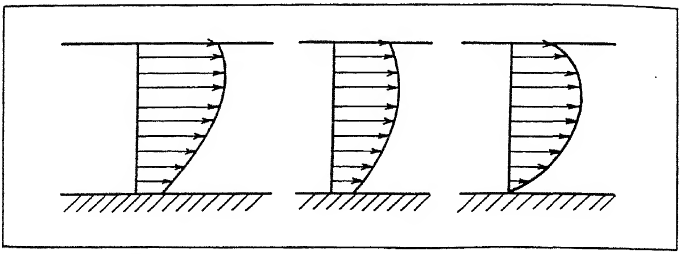

FIG. 155. Some vertical velocity profiles in open channels.

to bottom and from side to side. A bend in the channel or an irregular bed (as in a natural stream) may give very irregular and distorted velocity contour lines.

The measurement of the rate of discharge of a stream is frequently called *stream gaging*. A common procedure is to measure the velocity at various stations in the stream by means of a *current meter*. A current

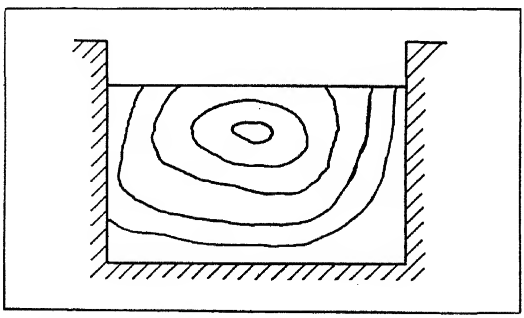

FIG. 156. Velocity contours in the straight length of an open channel.

meter in the stream may be suspended from a cable or attached to a rod. Measurements may be made from a bridge, a car suspended from a cable, or a boat. The actuating element of a current meter is a wheel, consisting of a series of vanes or cups, which is impelled by the stream current. The rate at which the wheel revolves varies with the water velocity. An indication of the rate of rotation gives a measure of the velocity if the meter is suitably calibrated. One procedure for cali-

brating or "rating" a current meter is to draw it through stationary water at a known speed.

If the velocity distribution in a stream varies, the actual kinetic energy of the stream is greater than that computed on the basis of average velocity. It is general practice, however, to express the kinetic energy term in the energy equation as $V^2/2g$, where V is the average velocity. This use of the average velocity, then, introduces an error. This error is sometimes compensated by empirical coefficients. In very accurate studies, as for laminar flow, it might be well to take the velocity variation into account. The general practice of using the average velocity is followed in this chapter.

125. Specific energy

The concept of *specific energy*, as developed by Bakhmeteff, has proved very fruitful in giving a simple and clear explanation of various phenomena of open-channel flow. An introduction to this concept, the specific-energy diagram, and some applications will be given in the following paragraphs.

There is a difference between *total head* and *specific energy*. The total head (total energy per unit weight of liquid) at any point in the stream shown in Fig. 154 is

$$\text{total head} = z + \frac{p}{w} + \frac{V^2}{2g},$$

where z is taken with respect to some *arbitrary horizontal datum plane*. The specific energy is defined as the energy per unit weight of the flowing liquid with respect to a line passing through the *bottom of the channel*. The specific energy E at any point, then, is

$$\text{specific energy} = E = d + \frac{p}{w} + \frac{V^2}{2g}. \tag{204}$$

Since $d + \frac{p}{w} = y$,

$$E = y + \frac{V^2}{2g}. \tag{205}$$

The specific energy in open-channel flow is simply the sum of the depth and the velocity head in the channel. In uniform open-channel flow the total head is decreased as flow takes place, but the specific energy remains constant. In nonuniform or varied flow the total head is continually decreased, but the specific energy may be increased or decreased.

It is most convenient and instructive to illustrate principles by dealing with the two-dimensional flow in a channel of rectangular cross section. In such a channel, with vertical side walls, the flow in parallel vertical planes is identical. The study of flow in a rectangular channel gives

simple equations, and helps to bring out fundamentals clearly. The same general method of attack can be applied to channels of other shapes, but the resulting relations are more complicated.

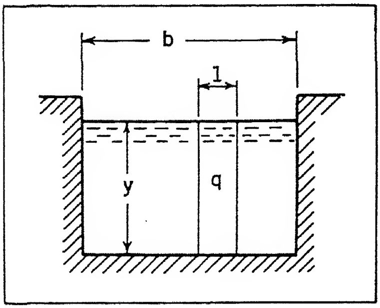

FIG. 157. Rectangular cross section of an open channel.

The total rate of discharge Q of the entire channel in Fig. 157 equals Vby. The rate of discharge q per unit width is simply Vy. For this two-dimensional flow, since $V = q/y$, the specific energy becomes

$$E = y + \frac{V^2}{2g} = y + \frac{q^2}{2gy^2}. \quad (206)$$

The foregoing equation involves essentially three variables. The physical aspects of Equation (206) can be brought out by treating separately two cases: one in which q is constant but E and y vary; and another in which E is constant but q and y vary.

Figure 158 shows a plot of depth against specific energy for a constant q. The curve is asymptotic to the horizontal axis and the 45° line marked "potential energy." The 45° line is simply a plot of the first term of

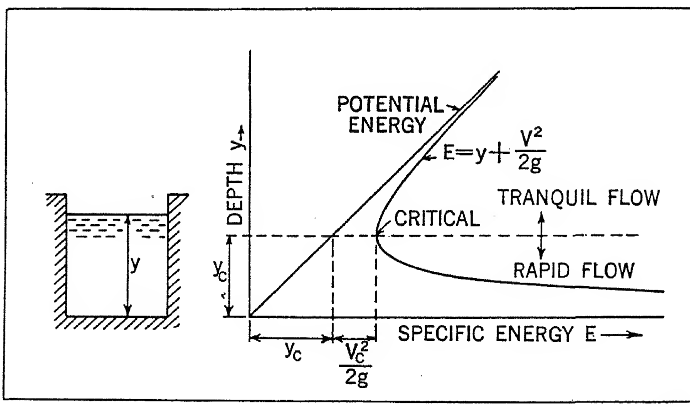

FIG. 158. Specific-energy diagram, with constant rate of discharge q.

Equation (206), whereas that part of the abscissa between the potential-energy line and the specific-energy curve represents the kinetic energy term $V^2/2g$ of Equation (206). The specific-energy curve has a point of minimum specific energy at a depth which is called the critical depth y_c. Considering values of depth other than y_c, for every specific energy

there are two alternate depths at which the flow may take place—one below and one above the critical depth. The flow is said to be *tranquil* if the actual depth y is greater than y_c. The flow is said to be *rapid* if the actual depth y is less than y_c.

For flow in the region of the critical depth, a depth change may correspond to a very small change in specific energy. Several depths may exist for practically the same specific energy. This feature offers some explanation of the fact that liquid flowing in the region of the critical depth may have an unstable, wavy surface. Figure 158 shows that a loss of specific energy is accompanied by a reduction of depth in tranquil flow, whereas a loss of specific energy is accompanied by an increase of depth in rapid flow.

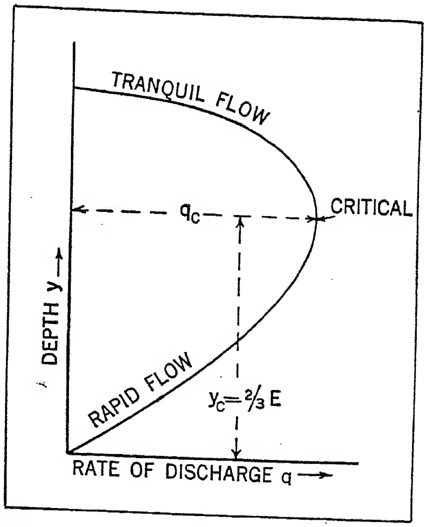

Fig. 159. Plot of depth against rate of discharge for a constant specific energy E.

Critical depth relations can be established by differentiating Equation (206) with respect to y and setting the result equal to zero. E has a minimum value when $dE/dy = 0$. The subscript c will be used for critical depth values. For example, E_c represents the specific energy at the critical depth y_c. Then

$$\frac{dE}{dy} = \frac{d}{dy}\left(y + \frac{q^2}{2gy^2}\right) = 1 - \frac{q^2}{gy^3}. \tag{207}$$

Setting Equation (207) equal to zero gives

$$q^2 = gy_c^3 \qquad \text{or} \qquad y_c = \sqrt[3]{\frac{q^2}{g}}. \tag{208}$$

Eliminating q from Equation (206) shows that

$$E_c = y_c + \frac{gy_c^3}{2gy_c^2} = \frac{3}{2}\,y_c. \tag{209}$$

Figure 159 shows a plot of depth y versus q for a constant value of specific energy E, as obtained by using Equation (206). Equation (206) can be rearranged as follows:

$$q^2 = 2g(Ey^2 - y^3). \tag{210}$$

The term dq/dy equals zero at the maximum value of q for a given E. Differentiating Equation (210) and setting $dq/dy = 0$ gives

$$\frac{dq}{dy} = \frac{g}{q}(2Ey - 3y^2), \qquad y_c = \frac{2}{3}E. \tag{211}$$

A comparison of Equations (209) and (211) shows that maximum rate of discharge and minimum specific energy both refer to conditions of critical flow.

The critical velocity V_c at the critical depth can be found from Equation (208) (noting that $q = yV$) to be

$$V_c = \sqrt{gy_c}. \tag{212}$$

An inspection of Equation (206) and Fig. 158 shows that at depths less than the critical the velocity $V > \sqrt{gy}$, whereas at depths greater than the critical the velocity $V < \sqrt{gy}$.

126. Nonuniform flow in channels of constant shape

The foregoing articles have dealt with uniform flow, a condition in which the depth, velocity, liquid cross-sectional area, slope, and other characteristics remain constant from section to section. In uniform flow the liquid surface is parallel to the bottom of the channel. Uniform flow might be regarded as a limit that is approached in some cases, as in very long channels. There is an unlimited number of ways in which a liquid might flow steadily through a certain channel. It might flow with a variable depth, and with a surface slope different from the bottom slope. In the most general case the energy line, the liquid surface, and the channel bottom have different slopes.

The following discussion is limited to nonuniform flow in a *prismatic* channel; the cross-sectional shape is constant along the channel axis in a prismatic channel. It will be assumed that the slope of the streamlines is constant, or varies so gradually that centrifugal forces are negligible. It will be assumed that the slope of the streamlines is so small that the pressure head does not differ appreciably from the depth. Such flow is sometimes called gradually varied flow. Because of a lack of a more satisfactory method, calculations are usually based on the assumption that the rate of energy loss in nonuniform flow is equivalent to that which would occur if the flow were uniform under identical conditions of depth, rate of discharge, and boundary roughness. This assumption has not been fully confirmed by experiment. Errors arising from such an assumption are likely to be small in comparison with those involved in the selection of roughness and friction coefficients.

The total head H at any point in the stream (see Fig. 160) is

$$H = z + y + \frac{V^2}{2g}, \tag{213}$$

where z is now the elevation of the channel floor. Differentiating with respect to distance in the direction of flow yields the corresponding rate of change of each of the variables:

$$\frac{dH}{dx} = \frac{dz}{dx} + \frac{dy}{dx} + \frac{d}{dx}\left(\frac{V^2}{2g}\right). \tag{214}$$

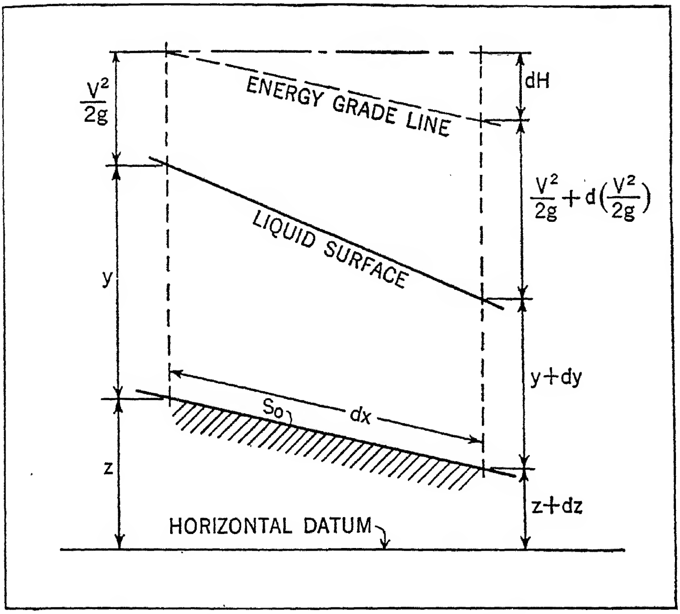

FIG. 160. Notation for nonuniform flow.

The term dH/dx represents the rate of change of total head. This quantity is always negative in the direction of flow. Using the Chezy notation gives

$$\frac{dH}{dx} = -\frac{V^2}{C^2R}.$$

The quantity dz/dx represents the slope of the bottom of the channel. Following the usual, arbitrary custom regarding signs (downward slope as positive), dz/dx will be taken as equal to $-S_0$.

The quantity dy/dx is the rate of change of depth with distance. It is to be recalled that $V = Q/A$, where A is the cross-sectional area of the liquid. If infinitesimals of order higher than the first are neglected, then an increment of cross-sectional area dA equals the surface width b multiplied by the depth increment dy (see Fig. 161). The last term of Equation (214) thus becomes

$$\frac{d}{dx}\left(\frac{V^2}{2g}\right) = \frac{V}{g}\frac{dV}{dx} = -\frac{Q^2}{A^3 g}\frac{dA}{dx} = -\frac{Q^2 b}{A^3 g}\frac{dy}{dx}.$$

Inserting these several equivalent terms in Equation (214), and solving for dy/dx, gives finally

$$\frac{dy}{dx} = \frac{S_0 - \dfrac{Q^2}{A^2C^2R}}{1 - \dfrac{Q^2 b}{A^3 g}}. \tag{215}$$

Equation (215) is the basic differential equation for the steady, non-uniform flow in a prismatic channel. Equation (215) can represent a large number of surface profile forms. A complete discussion of the various solutions of Equation (215) is beyond the scope of this book. An excellent treatment of methods for solving such equations is given by von Kármán and Biot.[1] Rouse[2] presents a discussion of various cases, and Bakhmeteff[2] has prepared tables which are applicable to any channel form.

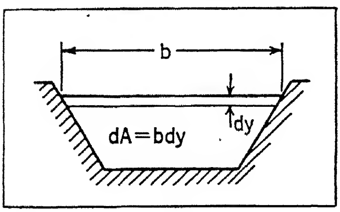

FIG. 161. Cross section of open channel.

If the denominator of Equation (215) is not zero, then $dy/dx = 0$ (uniform flow) when

$$S_0 = \frac{Q^2}{A^2C^2R} \qquad \text{or} \qquad A^2 = \frac{Q^2}{S_0C^2R}. \tag{216}$$

Therefore, for a particular form of channel, with a certain quantity and slope, there is only one depth at which the flow will be uniform. This depth is called the *normal depth.* When the actual bed slope is greater than the slope for uniform flow at a given depth, the numerator in Equation (215) is positive in sign, and vice versa. If the numerator in Equation (215) is not zero, then the slope dy/dx approaches infinity as the denominator of Equation (215) approaches zero. The limit is

[1] *Mathematical Methods in Engineering* by T. von Kármán and M. A. Biot. McGraw-Hill, New York, 1940.

[2] See references at end of this chapter.

reached at a critical depth y_c when

$$bQ^2 \qquad = 1. \qquad (217)$$

For a rectangular channel, $Q = byV$. A rearrangement of Equation (217) shows that the velocity V_c at the critical depth y_c is

$$V_c = \sqrt{gy_c}. \qquad (218)$$

Equation (218) is identical with Equation (212).

127. Hydraulic jump

The so-called *hydraulic jump* is an example of nonuniform flow in an open channel which will be studied by principles previously discussed.

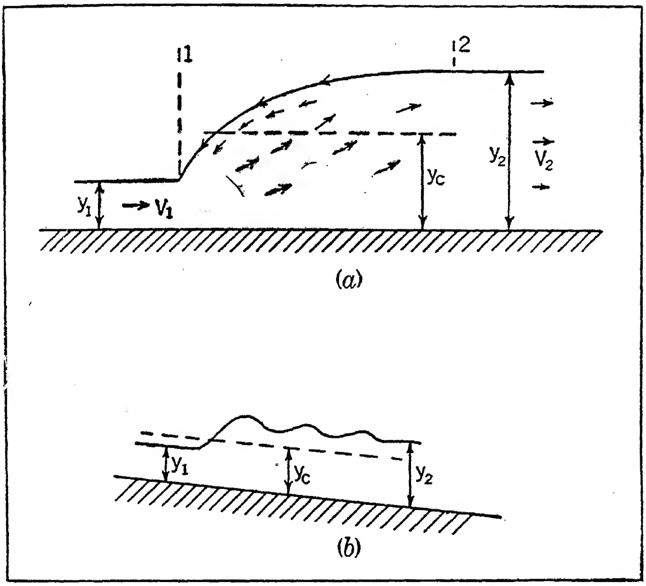

FIG. 162. Forms of hydraulic jump: (a) direct, (b) undular.

Under suitable circumstances the flow at a depth less than the critical may suddenly change to a flow at a certain depth greater than the critical. With this increase in depth there is a corresponding reduction in velocity. This curious local phenomenon, in which flow passes abruptly from a *rapid* to a *tranquil* state, is called the hydraulic jump. The hydraulic jump is somewhat similar to the sudden expansion in a pipe, in which there is a rapid increase in flow area, a corresponding decrease in velocity, and a loss of available mechanical energy.

The hydraulic jump may occur in different forms which may be classified broadly as (a) direct, and (b) undular. As illustrated in Fig. 162a, in the *direct* jump the upper depth is reached practically by one continuous rise of the surface. There is a surface roll and an eddying region at the face of the jump. The direct form is characteristic of jumps of relatively large height. In the *undular* form, Fig. 162b, the surface is wavy, but not broken, and there is no back roll. The undular form is typical of jumps of relatively low height.

Figure 163a indicates the profile of a jump, and Fig. 163b shows the corresponding plot of depth against specific energy. The subscript 1 refers to conditions before the jump, and the subscript 2 refers to conditions after the jump. At the depth $y_1(<y_c)$, the specific energy is E_1.

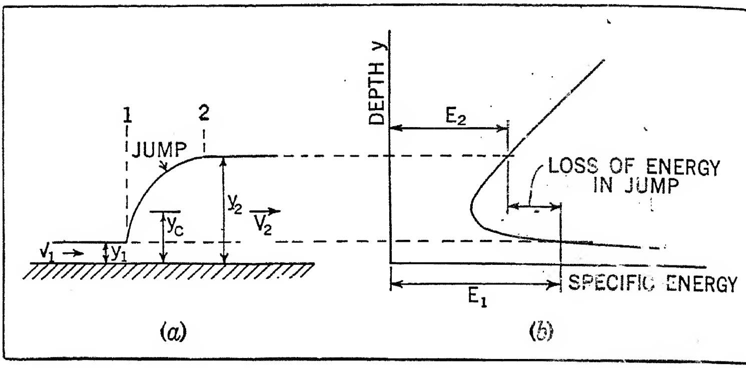

FIG. 163. Physical interpretation of the hydraulic jump by means of the specific-energy diagram.

At the depth $y_2(>y_c)$, the specific energy is E_2. $E_1 - E_2$ is the loss of specific energy inherent in the jump. This loss may be quite large under some circumstances.

In many problems in fluid mechanics and rigid-body mechanics an energy accounting alone does not provide a completely quantitative description of the phenomenon. Particularly in cases where the internal detailed mechanism is difficult to formulate, as when shock or impact is involved, it is very helpful to make a momentum study—a particular study or accounting based on conditions at the boundaries. The relation between the depths and velocities upstream and downstream from the hydraulic jump can be developed most conveniently by means of a momentum analysis.

Picture the body of liquid in the jump, Fig. 164, as a free or isolated body, with no tangential forces acting on the horizontal channel bottom and the side walls. It seems reasonable to neglect the tangential wall

forces if the length of the jump is short and the shock losses are relatively large. Then the net force in the direction of flow acting on the body of liquid is $F_1 - F_2$. Since force equals the time rate of change of linear momentum (Chapter 9), then

$$F_1 - F_2 = \frac{Qw}{g}(V_2 - V_1). \tag{219}$$

In order to demonstrate methods and principles with a minimum of mathematical complications, the case of two-dimensional flow will be

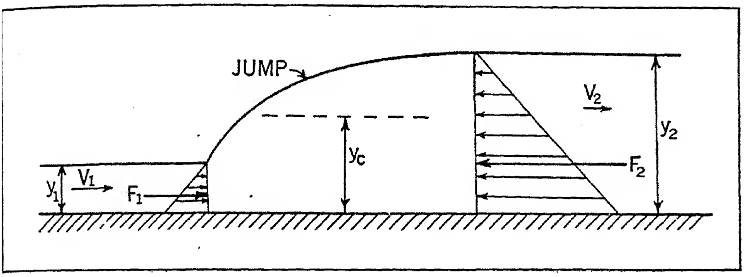

FIG. 164. Notation for momentum study of hydraulic jump.

treated, using a unit width of a rectangular channel, as shown in Fig. 157. Then, from principles of fluid statics (Chapter 2),

$$F_1 = \frac{wy_1}{2}A_1 = \frac{wy_1^2}{2}, \qquad F_2 = \frac{wy_2^2}{2},$$

From the equation of continuity, $V_1y_1 = V_2y_2 = q$. Substituting the foregoing relations in Equation (219) gives

$$\frac{y_1^2 - y_2^2}{2} = \frac{q^2}{g}\left(\frac{1}{y_2} - \frac{1}{y_1}\right),$$

$$\frac{y_1^2}{2} + \frac{q^2}{gy_1} = \frac{y_2^2}{2} + \frac{q^2}{gy_2}. \tag{220}$$

If the rate of discharge and one depth are known, the other depth can be calculated by Equation (220). Equation (220) can be solved explicitly, for y_1 and y_2, to give

$$y_2 = \frac{y_1}{2}\left[-1 + \sqrt{1 + \frac{8q^2}{gy_1^3}}\right],$$

$$y_1 = \frac{y_2}{2}\left[-1 + \sqrt{1 + \frac{8q^2}{gy_2^3}}\right]. \tag{221}$$

As an example, if the rate of discharge and y_1 are known, then y_2 can be calculated from the foregoing momentum equation. Then the loss

of energy, per unit weight, in the jump can be computed by the energy equation, as

$$\text{lost energy} = \left(y_1 + \frac{V_1^2}{2g}\right) - \left(y_2 + \frac{V_2^2}{2g}\right). \tag{222}$$

128. Flow over a fall

Figure 165 illustrates another example of nonuniform flow, that of the flow over a fall. With a horizontal channel bottom, the motion takes place entirely at the expense of the specific energy stored in the liquid. The passage from section 1 to section 2 on the falling liquid surface curve corresponds to a shift down on the upper branch (tranquil) of the specific energy curve, with a loss in specific energy. This loss in specific energy is accompanied by a lowering of the depth. The surface of the moving

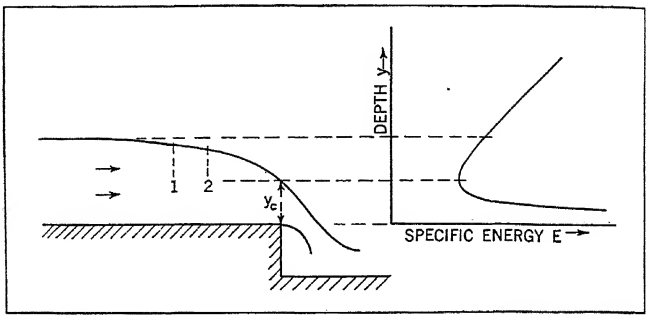

FIG. 165. Flow over a fall.

liquid cannot drop below the critical depth. The critical depth corresponds to the lowest possible energy content of the falling liquid. Any further lowering of the liquid surface below y_c would mean a change of the movement into the lower branch of the specific-energy curve—this change could be possible only if energy were added from outside. Therefore, the critical depth is the lowest depth which the surface can attain in the natural process of dissipating energy.

129. Analogy between open-channel flow and flow of compressible fluids

There are features associated with the propagation of a pressure wave through a compressible fluid which are somewhat similar to those involved in the travel of a slight disturbance at the free surface of a liquid. Some of these analogies will be discussed briefly in the following paragraphs.

By methods similar to those employed in Chapter 12, it can be shown that the velocity c_w of small-amplitude surface waves in a shallow open

channel is given by the expression[3]

$$c_w = \sqrt{gy}. \tag{223}$$

where y is the depth of the liquid. For example, if a stone were dropped into still water, ripples would move radially in all directions with the velocity c_w. Equation (223), however, could not be applied to ocean waves or waves in very deep channels. It is to be noted that the wave velocity c_w equals the velocity of open-channel flow at the critical depth y_c, as indicated by Equations (212) and (218):

$$V_c = \sqrt{gy_c}.$$

For critical flow, the liquid velocity equals the velocity of propagation of a small surface wave. If a wave were started, it could not progress upstream but would remain stationary because the two velocities are equal. In tranquil flow the liquid velocity is less than $\sqrt{gy}$. In tranquil flow small surface waves would travel upstream. In rapid flow the liquid velocity is greater than $\sqrt{gy}$. In rapid flow small surface waves would be swept downstream.

The phenomenon of the shock wave in a compressible fluid is somewhat analogous to the hydraulic jump occurring in open-channel flow. The following treatment of this analogy is similar to that given by von Kármán.[4]

The flow of a compressible fluid is supersonic if the fluid velocity V is greater than the velocity of pressure propagation c. The compressible flow is subsonic if V is less than c. The hydraulic jump is an abrupt transition between what might be called *supercritical* and *subcritical* flow. The flow of liquid in an open channel is supercritical if the liquid velocity V is higher than the velocity of surface waves c_w or $\sqrt{gy_c}$. The flow of liquid is subcritical if the velocity V is less than this critical value.

The continuity equation for steady flow in an open channel has the form

$$Vy = \text{constant}. \tag{224}$$

This relation is analogous to the continuity equation for the steady flow of a compressible fluid through a pipe of constant cross section,

$$V\rho = \text{constant}, \tag{225}$$

if the density ρ of the compressible fluid is taken as analogous to the depth y of the liquid flow.

Consider next the energy equation, in differential form without friction, for each flow. Assuming a horizontal channel bottom, the

[3] For a derivation see *Hydrodynamics* by H. Lamb. Cambridge University Press, Sixth Edition, 1932, page 254.

[4] *Problems of Flow in Compressible Fluids*, in the book *Fluid Mechanics and Statistical Methods in Engineering*. University of Pennsylvania Press, 1941, page 15.

total head for open-channel flow equals $y + (V^2/2g)$. Differentiating this relation gives

$$gdy + VdV = 0. \tag{226}$$

The energy equation for the flow of a compressible fluid is (see Equation 164):

$$gvdp + VdV = 0 \quad \text{or} \quad \frac{dp}{\rho} + VdV = 0. \tag{227}$$

Recalling that $dp/d\rho = c^2$, Equation (227) can be written as

$$\frac{dp}{d\rho}\frac{d\rho}{\rho} + VdV = 0,$$

$$c^2\frac{d\rho}{\rho} + VdV = 0. \tag{228}$$

c^2 corresponds to gy. If ρ is replaced by y, then $d\rho/\rho$ corresponds to dy/y. Thus $c^2(d\rho/\rho)$, or $(dp/d\rho)(d\rho/\rho)$, corresponds to gdy, and Equations (226) and (228) are analogous.

Thus both the continuity equation and the energy equation for compressible flow are analogous to those for open-channel flow if y and ρ are regarded as corresponding quantities.

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Fluid Mechanics for Hydraulic Engineers by Hunter Rouse. McGraw-Hill, New York, 1938.

Hydraulics of Open Channels by B. Bakhmeteff. McGraw-Hill, New York, 1932.

Applied Fluid Mechanics by M. P. O'Brien and G. H. Hickox. McGraw-Hill, New York, 1937.

PROBLEMS

130. A circular, brick-lined conduit, 5 feet in diameter, has a slope of 1 in 1000. Compute the rate of discharge for steady, uniform flow if the conduit flows half full.

131. A rectangular channel 25 feet wide and 6 feet deep is lined with unplaned wood. The rate of discharge is 400 cubic feet per second. What is the slope if the flow is steady and uniform?

132. A canal, lined with unfinished concrete, is of trapezoidal section with a bottom width of 10 feet and sides making an angle of 60° with the horizontal. The bottom slope is 0.0015 and the depth of flow is 7 feet. Calculate the rate of discharge for steady uniform flow.

133. Sometimes for practical purposes it is important to proportion the dimensions of the channel cross section to give a hydraulic mean depth which is as large as possible. A rectangular cross-section has a width b and a depth d. For a given area A, find the ratio between b and d to give a maximum hydraulic mean depth. (*Hint:* Express R in terms of A and d, and then differentiate with respect to d.)

134. A trapezoidal canal has a bottom width of 20 feet and side slopes of 1 to 1. The rate of discharge is 364 cubic feet per second at a depth of 3.60 feet. What is the specific energy?

135. If 600 cubic feet per second at a depth of 4.1 feet flows in a rectangular channel 20 feet wide, is the flow rapid or tranquil?

136. What is the maximum rate of flow which could occur in a rectangular channel 16 feet wide for a specific energy of 6.8 feet?

137. Starting with Equation (206), for a constant rate of discharge derive the equation

$$\frac{E}{y_c} = \frac{1}{2}\left(\frac{y_c}{y}\right)^2 + \frac{y}{y_c},$$

in which all the ratios are dimensionless. Plot a dimensionless specific energy diagram with y/y_c as ordinates and E/y_c as abscissae.

138. Starting with Equation (210), for a constant specific energy derive the relation

$$\frac{q}{q_c} = \sqrt{3\left(\frac{y}{y_c}\right)^2 - 2\left(\frac{y}{y_c}\right)^3},$$

where q_c is the critical discharge, and each ratio is dimensionless. Plot a dimensionless diagram with y/y_c as ordinates and q/q_c as abscissae.

139. Derive Equation (221) from Equation (220).

140. Water flows 3 feet deep in a rectangular channel with an average velocity of 4 feet per second. Can a jump be formed downstream?

141. Water flows at a rate of 360 cubic feet per second in a rectangular channel 18 feet wide at a depth of 1.0 foot. What is the total energy loss in a hydraulic jump which has occurred from this flow?

142. With a discharge of 400 cubic feet per second in a rectangular channel 20 feet wide, determine the depth y_1 which will sustain a jump resulting in $y_2 = 4.0$ feet. What portion of the initial energy is lost in the jump?

CHAPTER 15

Lubrication

A very important problem that confronts many engineers is that of the proper lubrication of the moving parts of a machine. The use of a lubricant serves several purposes: (a) to make motion between mechanical parts possible; (b) to reduce the wear of these parts; and (c) to reduce the friction between these parts. This chapter discusses some lubrication fundamentals of practical importance. The fluid flow in the lubrication process is of the laminar or viscous type; the speeds may be high but the channel dimensions, in current practice, are so small that the Reynolds' numbers are low.

130. Mechanism of film lubrication

The lubrication process can be conveniently illustrated by referring to the lubricant film pressures developed in a slipper bearing. Figure 166

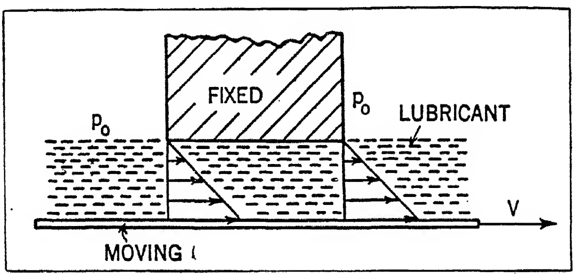


FIG. 166. Shearing of lubricant between parallel sliding surfaces. Action cannot sustain a load.

shows two parallel, sliding surfaces separated completely by a fluid lubricant; the flow is two-dimensional. A very thin layer of fluid adheres to the fixed surface, and has zero velocity. A very thin layer of fluid adheres to the surface of the moving slipper and has the velocity V. For small distances between the surfaces, the velocity gradient is linear and the flow is laminar. This velocity gradient is the same at all points along the channel length for parallel surfaces. The volume of lubricant drawn in by viscous drag equals that discharged. Therefore, no change in pressure, from p_0, results from the motion of the moving surface. The lubricant film itself cannot sustain a load.

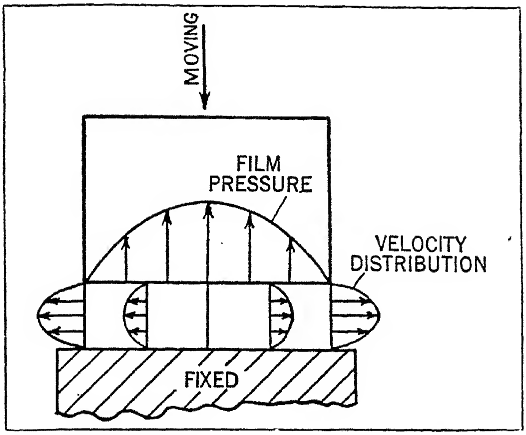


FIG. 167. Lubricant film flow and pressure between approaching surfaces.

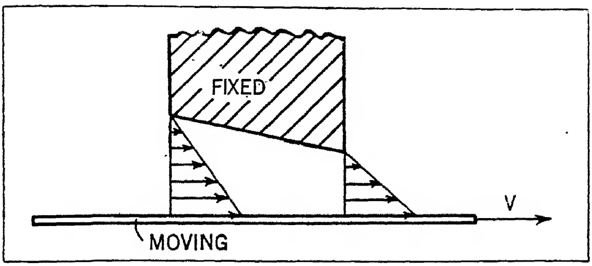


FIG. 168. Tendency of viscous drag alone for inclined surfaces.

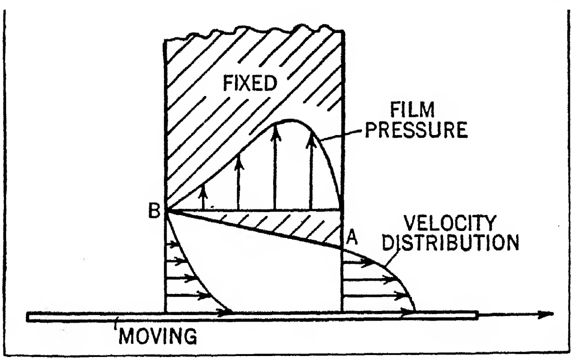


FIG. 169. Resultant flow of lubricant caused by film pressure and viscous drag. Converging-wedge action *can* sustain a load.

The resultant flow for inclined surfaces can be visualized as the combination of two separate flows. When two surfaces approach each other, as shown in Fig. 167, the lubricant is squeezed out from the space. Since a force is required to move the viscous fluid, a film pressure is developed as indicated. The flow, for a certain position of the moving body, is represented by the parabolic velocity profiles. When two sliding surfaces form a wedge, as shown in Fig. 168, the velocity triangles indicate that viscous drag tends to draw more lubricant into the space than it can take out. This tendency develops a film pressure. The flow due to the film pressure alone, in this way added to the flow due to the viscous drag, gives the resultant flow and pressure distribution shown in Fig. 169. The important result is that the action of the converging fluid wedge develops a film pressure which can sustain a load. In some respects the action is analogous to that of an airplane taking off from the ground.

131. Cylindrical or journal bearing

Figure 170a shows diagrammatically a cylindrical shaft or journal at rest in a bearing; the clearance is exaggerated for the sake of illus-

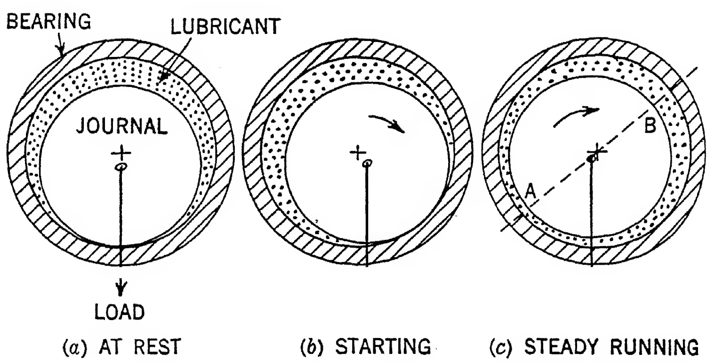


Fig. 170. Cylindrical or journal bearing.

tration. The space between journal and bearing is filled with an ample supply of lubricant. There is metal-to-metal contact at the lowest point, and the journal is loaded.

As the journal begins to rotate (see Fig. 170b), it crawls up the bearing to the right for the clockwise rotation indicated. Fluid adheres to the surface of the journal, and the journal rolls on a lubricated surface. If sufficient speed is developed, the lubricant will be drawn between the surfaces, to separate the surfaces and float the journal, as shown in Fig. 170c. The shaft center moves to the left side of the bearing center. In Fig. 170c there is a converging film from B to A in the direction of

rotation. If the speed is high enough, and if there is an adequate supply of lubricant, this viscosity pump action develops a positive pressure and supports the load. There is also a diverging film in which the pressure is lower than that in the converging film. Proper clearance must be provided for the pump action to develop.

Figure 171 shows the general variation of the radial or polar pressure distribution as indicated by experimental measurements. The maximum fluid pressure is reached at a point on one side of the line of loading. It is generally recommended that the oil feed to the bearing be introduced in the diverging section. If an oil groove is cut in the bearing in the

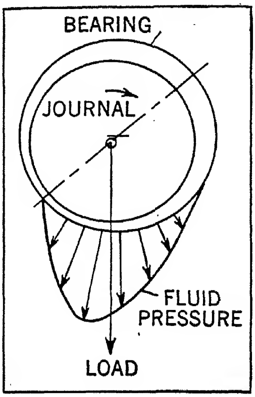


Fig. 171. Pressure distribution around the circumference of a bearing—radial ordinate represents positive pressure.

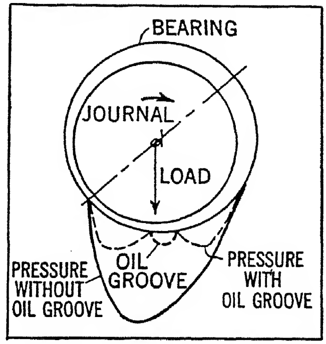


Fig. 172. Effect of oil groove on film pressure.

positive-pressure region, the pressure at the groove will be the pressure at which oil is supplied to the bearing. The pressure distribution will be somewhat like that shown by the dotted line in Fig. 172. Figure 172 indicates how a groove in the positive-pressure region causes a large reduction in the carrying capacity of a bearing, as compared with a bearing that has no groove.

In many lubrication systems an external pump is employed to force lubricant to the bearing. The function of this feed pump is to overcome friction in the feed pipe and fittings, to *provide an adequate supply* of lubricant. After lubricant is supplied to the bearing, the viscosity pump action itself develops fluid pressure to support the journal.

132. Journal-bearing friction

The following notation will be used in discussing journal friction: N = journal revolutions per unit time; μ = dynamic viscosity of the

fluid lubricant; D = journal diameter; L = length of bearing along axis of rotation; C = diametral clearance, difference between diameter of bearing and diameter of journal; f = friction coefficient, dimensionless ratio of friction force R (tangent to rotating journal) to journal load P, $f = R/P$; p = projected bearing pressure, total journal load divided by projected bearing area, $p = P/LD$.

Consider a series of geometrically similar journal bearings, all having the same clearance-to-diameter ratio, and the same length-to-diameter ratio. If the only variables involved are f, μ, N, and p, then dimensional analysis[1] shows that the form of the physical equation can be expressed in terms of two dimensionless ratios, f and $\mu N/p$. It is common practice to correlate measurements in terms of these ratios. Then the physical equation for bearing friction can be expressed as

$$f = \text{some function of } \frac{\mu N}{p}.$$

Figure 173 shows a typical plot of journal-friction measurements; any set of consistent units can be employed for the dimensionless ratio

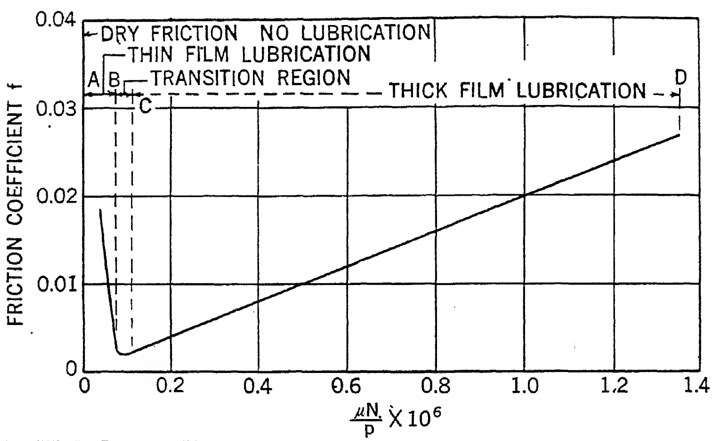


FIG. 173. Typical plot of journal-friction measurements. $L/D = 1.00$ and $C/D = 0.001$.

So-called *perfect* or *thick-film* lubrication exists in the region from C to D; in this region the viscosity pump action is fully developed. With thick-film lubrication, the sliding surfaces are completely separated by a lubricant—there is no metal-to-metal contact, and the wear is minimized. The phenomenon involved in thick-film lubrication is

[1] See Example 1, Article 46.

primarily one of viscous flow; the frictional resistance is due mainly, if not completely, to the shearing of the lubricant. There are indications that the frictional losses in the region CD are independent of the bearing materials. The oil film becomes thinner as operation proceeds from D to C.

So-called *imperfect* or *thin-film* lubrication occurs in the region AB. In thin-film lubrication there may be some metal contact, wear, or seizure. With thin-film lubrication the film thickness may be so small that other forces besides viscous are involved—the frictional loss depends upon the character and materials of the bearing surfaces, the physical-chemical nature of the lubricant, and on the unknown intermolecular forces between the fluid and solid surfaces.

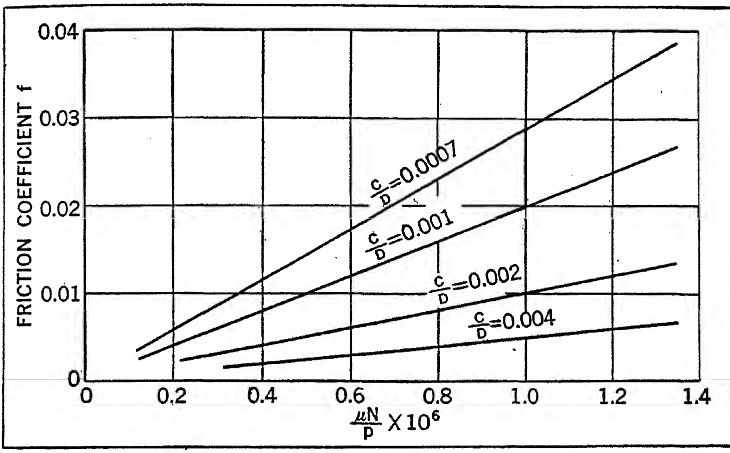


FIG. 174. Effect of clearance on thick-film lubrication.

Because nearly all the wear of a bearing with thick-film lubrication under rated operating conditions may take place during the starting period, a machine should be operated, whenever possible, to start under no load or a light load. A cold, fast start of an automobile engine may damage the unlubricated parts more than a day of driving when the motor is warm.

The region BC in Fig. 173 marks a transition between stable and unstable lubrication. The location of this transition region may be different for different bearing arrangements. Stable lubrication exists in the region CD. As the friction is increased, the lubricant temperature rise reduces the viscosity, and hence there is a tendency to reduce the friction. The region AB is one of unstable lubrication. An increase in temperature reduces the viscosity. Since the curve slopes upward to the left, a decrease in viscosity causes an increase in friction.

Modern machine design books give recommended values of $\mu N/p$ and other factors for different classes of machinery. The general effect of different clearance-to-diameter ratios is indicated in Fig. 174. In ordinary machine-shop practice it is common to use a ratio $C/D = 0.001$. For example, for a shaft of 2-inch diameter the diametral clearance would be 0.002 inch.

133. Thin-film friction

Some of the factors affecting the position of the minimum point in Fig. 173 are: (a) roughness of the surfaces; (b) materials composing the surfaces; (c) constitution and properties of the lubricant; and (d) amount

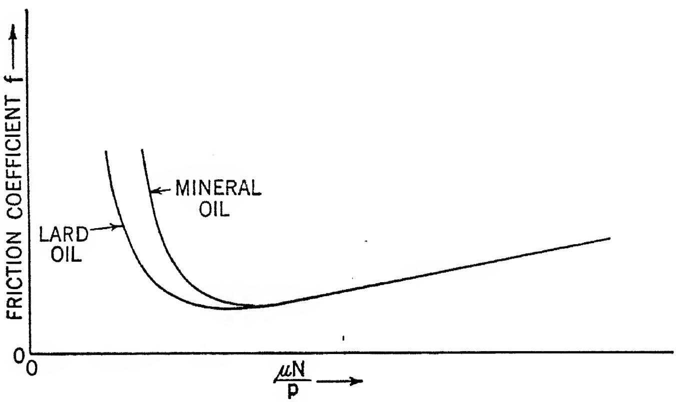


FIG. 175. Effect of oiliness on friction coefficient for a journal bearing.

of oil supply. In the thin-film region different oils may give different friction coefficients for the same value of $\mu N/p$. This possibility is illustrated in Fig. 175. Two oils may have the same dynamic viscosity at a certain temperature and pressure, but they may differ in chemical properties. The term *oiliness* is sometimes employed in discussing the frictional characteristics (as distinguished from wear and seizure characteristics) of a lubricant or a lubricant-metal combination.

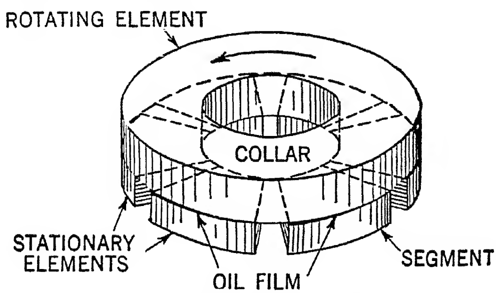
According to Hersey,[2] "Oiliness is that which may cause a difference in friction between two tests, under identical conditions, in which the lubricants have the same viscosity at the test temperature and atmospheric pressure." The nature of this somewhat elusive property of "oiliness" has not been fully established at the present time.

[2] *Logic of Oiliness* by M. D. Hersey, *Mechanical Engineering*, September, 1933, page 561.

134. Kingsbury thrust bearing

An interesting and practically important application of the fluid-wedge lubricating action is found in the thrust bearing developed by A. Kingsbury. As indicated in Fig. 176, the Kingsbury bearing makes possible the automatic formation of wedge-shaped oil films under a thrust load (along the axis of rotation), accomplishing in thrust bearings what a well-designed journal bearing does for radial loads. One of the bearing elements is divided into segments, usually three or six in number. These segments are pivoted and supported so that they are free to tilt slightly. The oil films assume automatically whatever taper is required by the speed, load, and lubricant viscosity.

The friction coefficient for a Kingsbury thrust bearing varies approximately from 0.001 to 0.005, depending on the load, speed, and viscosity.



Courtesy of Kingsbury Machine Works, Inc.

FIG. 176. Basic elements of the Kingsbury thrust bearing, showing the wedge-shaped oil films.

Such a bearing will easily sustain loads of 300 pounds per square inch of segment area. Kingsbury thrust bearings are employed in a wide variety of applications, some on very large machinery, as on pumps, propeller shafts on marine vessels, and hydroelectric installations.

135. Viscosity index

From one point of view, the ideal oil is one which has the same dynamic viscosity at all temperatures. No such oil has been found. In general, the dynamic viscosity of a liquid decreases as the temperature increases. The rate of change of viscosity with temperature is different for different liquids, and may be an important physical factor, particularly for lubricants.

Lubricating oils are subjected to a wide range of temperatures in service, which inevitably means a wide fluctuation in dynamic viscosity. The viscosity of an oil may drop to a value where the lubricating film is broken, resulting in metal-to-metal contact and excessive wear. On the other extreme, the viscosity may rise to a value where the oil becomes

too viscous for proper circulation, and the bearing may run dry. In an automobile, for example, the crankcase oil may be required to flow freely at 0° Fahrenheit at starting, and also to have sufficient viscosity to provide safe lubrication at 300° Fahrenheit when operating. Therefore, much attention has been paid, in recent years, to relationships between viscosity and temperature. One relationship that has been developed is the so-called *viscosity index.* Although empirical, this index has certain advantages for some purposes, and is in common use.

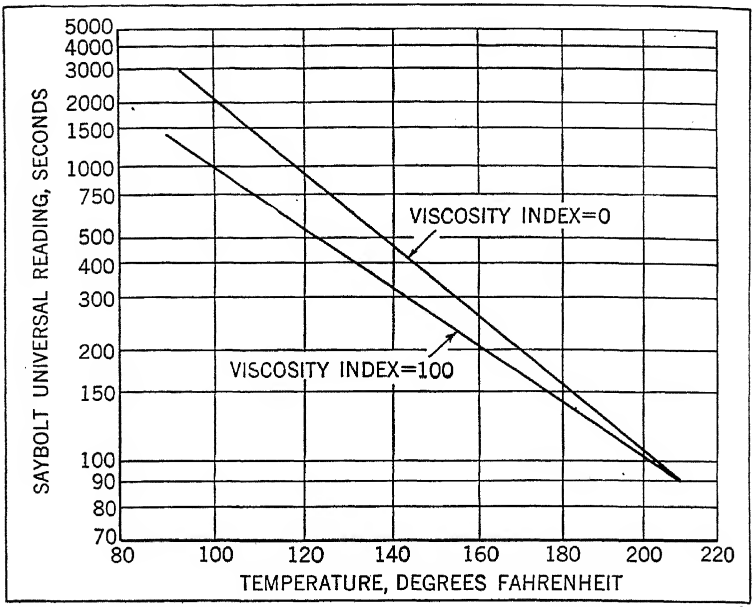


Fig. 177. Saybolt reading plotted against temperature for two oils of different viscosity indexes.

According to the A.S.T.M. (American Society for Testing Materials) Standards,[3] "The viscosity index is an empirical number indicating the effect of change of temperature on the viscosity of an oil. A low viscosity index signifies relatively large change of viscosity with temperature." The viscosity index of an average naphthene oil has arbitrarily been assumed 0, whereas the viscosity index of an average paraffin oil has arbitrarily been taken as 100. The viscosity index of an oil can be determined from the relation

$$\text{viscosity index (V.I.)} = \frac{(L - U)}{(L - H)}\,100,$$

[3] *Standard Method for Calculating Viscosity Index,* A.S.T.M. Standard D567-41.

where U is the Saybolt reading at 100° Fahrenheit of the oil whose viscosity index is to be calculated; L is the Saybolt reading at 100° Fahrenheit of an oil of 0 viscosity index having the same Saybolt reading at 210° Fahrenheit as the oil whose viscosity index is to be calculated; and H is the Saybolt reading at 100° Fahrenheit of an oil of 100 viscosity index having the same Saybolt reading at 210° Fahrenheit as the oil whose viscosity index is to be calculated.

As an example, consider the change in Saybolt reading of two oils, one having a viscosity index of 100, and the other with a viscosity index of 0, both having the same Saybolt reading of 90 seconds at 210° Fahrenheit. The oil with a viscosity index of 100 would have a Saybolt reading of 986 seconds at 100° Fahrenheit,[4] whereas the oil with a viscosity index of 0 would have a Saybolt reading of 2115 seconds at 100° Fahrenheit. These results are shown graphically in Fig. 177.

An oil of high viscosity index does not become excessively viscous when cooled, or excessively thin when heated. An oil of high viscosity index is usually desirable for services in which the oil is exposed to wide variations in temperature. The A.S.T.M. has standardized various charts[5] which are so constructed that for any given petroleum oil the viscosity-temperature points lie on a straight line within normal ranges.

136. Properties of petroleum lubricating oils

Figure 178 shows the kinematic viscosity of various types of petroleum lubricating oils at different temperatures. Figure 179 shows the density of these same oils at different temperatures. The dynamic viscosity of each lubricant can be determined from the kinematic viscosity and the density. It is to be recalled that one stoke equals one centimeter squared

TABLE 9

DESCRIPTION OF VARIOUS TYPES OF PETROLEUM LUBRICATING OILS WHOSE CHARACTERISTICS ARE PLOTTED IN FIGURES 178 AND 179

Legend Letter	*Type of Oil*
A	High-viscosity-index S.A.E. 10 motor oil
B	High-viscosity-index S.A.E. 30 motor oil
C	High-viscosity-index S.A.E. 50 motor oil
D	Low-viscosity-index S.A.E. 10 motor oil
E	Low-viscosity-index S.A.E. 30 motor oil
F	High-viscosity-index S.A.E. 60 aviation engine oil
G	Transformer oil
H	Gear oil, S.A.E. 90
I	Gear oil, S.A.E. 140
J	Steam-turbine oil, light
K	Steam-turbine oil, medium

[4] Such data can be found in tables given in *Standard Methods for Calculating Viscosity Index*, A.S.T.M. Standard, D567-41.

[5] *Standard Viscosity-Temperature Charts for Liquid Petroleum Products*, A.S.T.M. Standard D341-39.

per second. The legend for Figures 178 and 179, with a description of each oil, is listed in Table 9. These data give a general picture of the characteristics of common lubricating oils.

The oils *A* and *D* in Fig. 178 are both designated as S.A.E. 10 motor oils (Society of Automotive Engineers designation). Oil *A*,

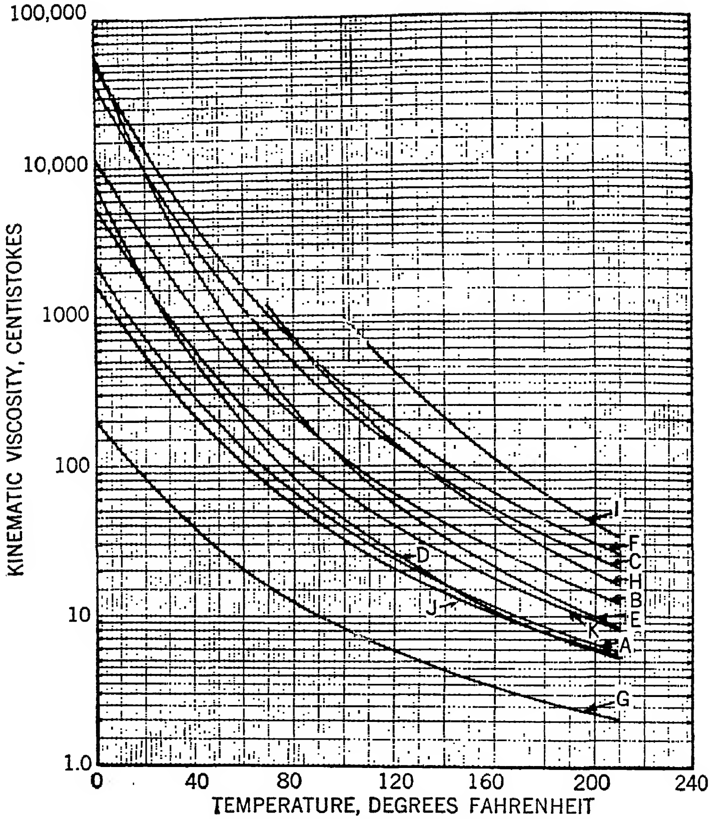


Fig. 178. Kinematic viscosity of various types of petroleum lubricating oils as a function of temperature. For legend and description of types, see Table 9 in text. (Curves plotted from data furnished through courtesy of Gulf Research and Development Co.)

however, has a higher viscosity index than oil *D*; the Saybolt reading and the kinematic viscosity of oil *A* change less with temperature than oil *D*. A similar comparison applies to oils *B* and *E*. For practical problems the specific gravity is taken as numerically equal (not dimensionally equal) to the density in grams per milliliter. The mass of one milliliter

of pure water at 4° centigrade (39.2° Fahrenheit) is one gram. Figure 179 shows that, for all the oils plotted, the density (and therefore the specific gravity, and specific weight) decreases almost linearly with an increase in temperature.

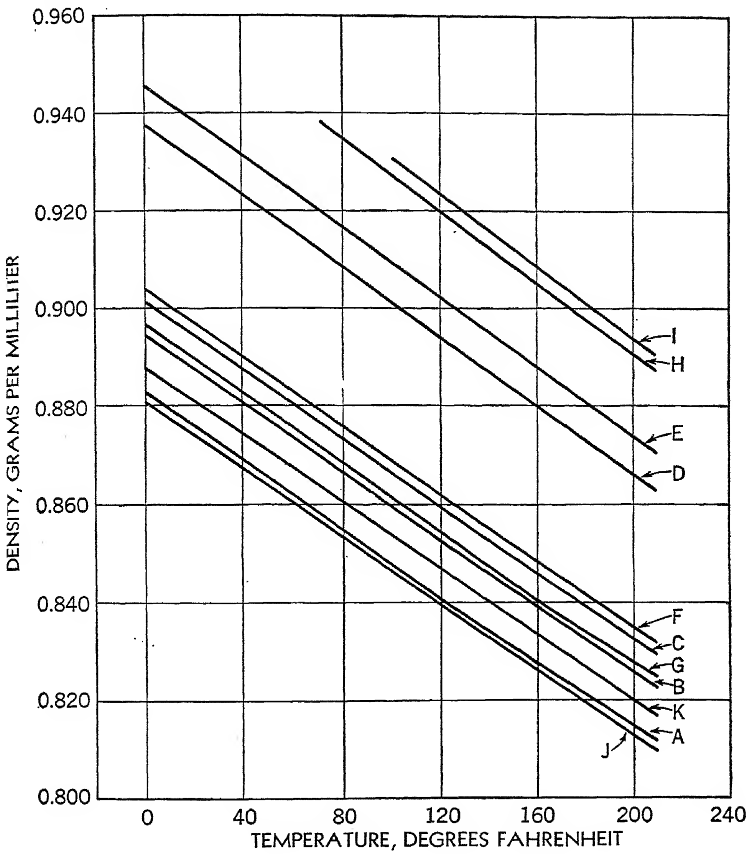


FIG. 179. Density of various types of petroleum lubricating oils as a function of temperature. For legend and description of types, see Table 9 in text. (Curves plotted from data supplied through courtesy of Gulf Research and Development Co.)

Two "gravity" scales are in general use in the petroleum industry, the specific gravity scale and the A.P.I. (American Petroleum Institute) gravity scale. The relation between the A.P.I. gravity and the specific gravity is given by the empirical relation

$$\text{A.P.I. degrees at } 60^\circ \text{ F.} = \frac{141.5}{\text{sp. gr. at } 60^\circ \text{ F.}} - 131.5.$$

The A.P.I. gravity varies inversely as the specific gravity. A low A.P.I. gravity means a high specific gravity; whereas a high A.P.I. gravity indicates a low specific gravity.

137. Theoretical relations for two-dimensional flow in a journal bearing

The foregoing articles have emphasized the physical features of lubrication. A great deal of mathematical work has been done in the field of thick-film lubrication, and the reader interested in this aspect can find abundant material in the literature. Viscous or laminar flow, in general, is more amenable to mathematical analysis than turbulent flow; purely mathematical investigations have proved very fruitful. A comprehensive treatment of such complicated work, however, is difficult to summarize adequately in one chapter. Further, theoretical results should be applied with caution, because of limitations of the theory. Accordingly, only a brief, simplified analytical treatment will be given in this article and the next, in order to shed further light on the thick-film lubrication process, and to give some idea of general methods of approach.

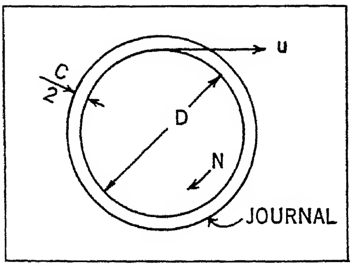


FIG. 180. Notation for a cylindrical bearing.

The problem of two-dimensional flow in a cylindrical or journal bearing will be studied first. Two-dimensional flow may be realized in a bearing of very long length (length along the axis of rotation). It will be assumed that the shaft is lightly loaded and that the speed is so high that the eccentricity of shaft center and bearing center is small. It will be assumed that the radial clearance $C/2$ is so small in comparison with the shaft diameter D that the curvature of the path of the lubricant can be neglected. As shown in Fig. 180, let u be the peripheral velocity of the surface of the rotating shaft.

Dynamic viscosity is the ratio of shear stress to rate of shearing strain (Chapter 5). Taking the rate of shearing strain at the shaft surface as $2u/C$, the shear stress R/A at the shaft is then

$$\frac{R}{A} = \mu \frac{2u}{C}, \tag{229}$$

where A is the surface area over which the shear force R acts. The velocity $u = \pi DN$, and the area of the cylindrical surface is πDL. Thus

the friction force becomes

$$R = \mu(\pi DL)\,\frac{2\pi DN}{C}$$

The journal load $P = pLD$. Therefore, the friction coefficient can be expressed as

$$f = \frac{R}{P} = \frac{2\mu(\pi DL)\pi DN}{pLDC},$$

$$f = 2\pi^2\left(\frac{D}{C}\right)\frac{\mu N}{p}. \tag{230}$$

Equation (230) is commonly called the Petroff equation. Equation (230) shows that the friction coefficient is independent of the length L.

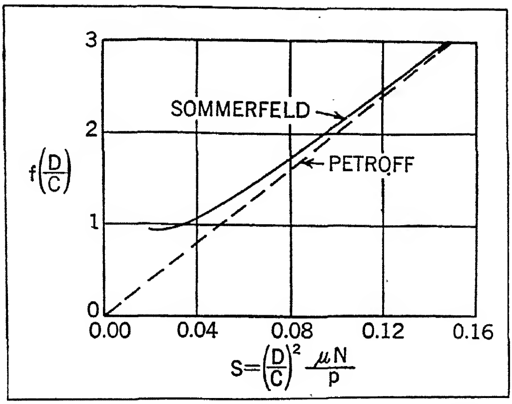


FIG. 181. Theoretical variation of friction coefficient for journal bearings.

If f is plotted against $(D/C)(\mu N/p)$, the resulting curve is a straight line passing through the origin with a slope equal to $2\pi^2$.

Sommerfeld's classical work[6] presents a more refined analysis of this particular problem. He expressed his theoretical results in the form

$$f\left(\frac{D}{C}\right) = \text{some function of}\left[\left(\frac{D}{C}\right)^2\frac{\mu N}{p}\right]. \tag{231}$$

The dimensionless ratio $(D/C)^2(\mu N/p)$ is usually called the Sommerfeld variable and is commonly denoted by the letter S. The product $f(D/C)$ is also a dimensionless ratio.

[6] *Zur hydrodynamischen Theorie der Schmiermittelreibung* (*On the Hydrodynamic Theory of Lubrication*) by A. Sommerfeld, *Zeit. Math. Phys.*, Bd. 50, 1904, pages 97–155.

Petroff's equation is shown by the dotted line in Fig. 181, and Sommerfeld's results are represented by the full line in Fig. 181. The Sommerfeld curve approaches asymptotically the Petroff relation. Recent experiments on thick-film lubrication by Morgan and Muskat[7] show results which agree fairly well with these theoretical relations; their values (plotted as in Fig. 181) fall on a straight line which, when extended, passes through the origin. The curves in Fig. 181 apply only for thick-film lubrication. At low values of $(D/C)^2(\mu N/p)$ there is a marked increase in $f(D/C)$ for thin-film lubrication.

The life of a bearing is closely related to the minimum film thickness. Let the minimum film thickness in a bearing be denoted by h, and the radial clearance by $c = C/2$. The dimensionless eccentricity ratio h/c can be expressed as a function of the Sommerfeld factor, as was done

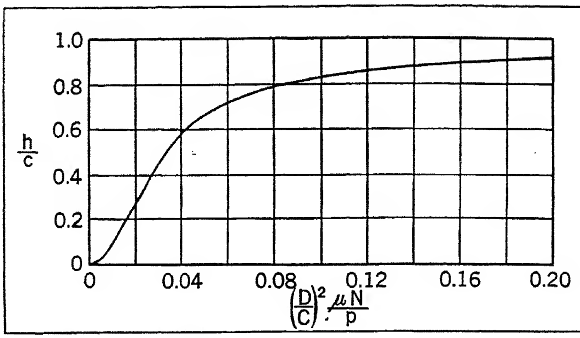


FIG. 182. Theoretical variation of minimum film thickness.

theoretically by Sommerfeld. Such a plot is shown in Fig. 182. The rotating shaft runs concentric with the bearing when $h/c = 1$. The shaft touches the bearing when $h/c = 0$. For a given clearance, the minimum film thickness increases with increasing viscosity, increasing journal speed, and decreasing load. Considerable wear of the bearing may occur during starting and stopping. For normal, safe operation, the journal should not operate below some limiting value of S, with a corresponding value of h/c.

The shearing stresses and the fluid pressures in the film tend to increase as either μ or N increases. As μ or N increases, a smaller wedging of the film will develop the resultant pressure to support the load on the journal. Thus there is a decrease in the journal eccentricity as $\mu N/p$ increases. On the other hand, as the journal load is decreased, keeping

[7] *Experimental Friction Coefficients for Thick Film Lubrication of Complete Journal Bearings* by F. Morgan and M. Muskat, *Journal of Applied Physics,* vol. 9, No. 8, August, 1938, page 539.

μN constant, a smaller fluid film pressure will support the load; this situation again means a smaller journal eccentricity.

138. Theoretical relations for two-dimensional flow in a slipper bearing

The two-dimensional flow of lubricant between a moving slipper and a plane guide, as represented in Fig. 169, will be investigated next. A treatment of this problem brings out mathematically some features which were discussed previously from a physical point of view, and also illustrates a general method of approach used in attacking viscous or laminar flow problems.

A fundamental relation for laminar flow will be derived first. Figure 183 shows the notation; u is the velocity at a distance y from some

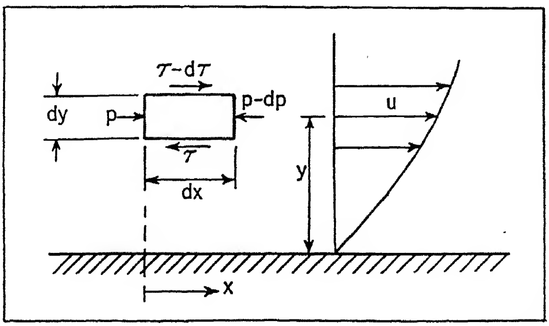


FIG. 183. Notation for two-dimensional laminar flow.

boundary, and x is a distance measured in the direction of flow. Consider the forces acting on a small element of height dy, of length dx, and of unit depth perpendicular to the plane of motion. The only forces acting on this element are the pressure and viscous shear forces (*no* inertia forces). For steady flow the pressure forces on this element balance the viscous forces. Therefore,

$$dp\,dy = d\tau\,dx$$

or

$$\frac{dp}{dx} = \frac{d\tau}{dy}$$

where $d\tau$ is the increment of the shear stress τ. Since $\tau = \mu\, du/dy$, the pressure gradient dp/dx becomes

$$\frac{dp}{dx} = \mu \frac{d^2u}{dy^2}. \qquad (232)$$

The pressure gradient will be regarded as constant in the direction of y because the film is very thin. Integration of the foregoing fundamental equation gives

$$u = \frac{1}{\mu}\left(\frac{dp}{dx}\right)\frac{y^2}{2} + C_1 y + C_2. \tag{233}$$

The constants C_1 and C_2 are to be evaluated from boundary conditions.

Equation (233) will now be applied to the inclined slipper bearing shown in Fig. 184. The upper body is fixed, whereas the lower plate moves with the velocity V. The origin of the coordinate system is taken at 0, with the y-axis along the left edge of the fixed body. It is assumed that the angle δ (Greek letter delta) is small. The film thick-

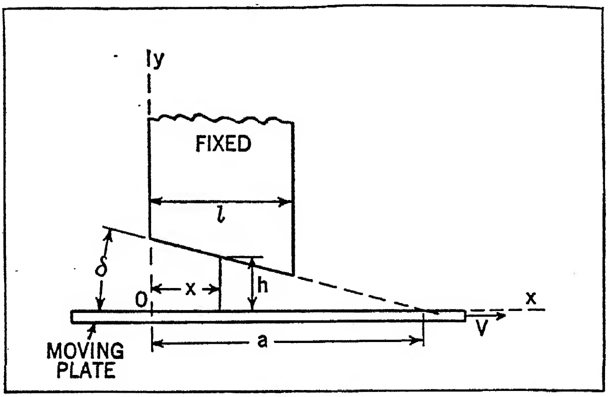


FIG. 184. Notation for slipper bearing.

ness is h at some distance x from the origin. Accordingly, $u = V$ for $y = 0$, and $u = 0$ when $y = h$. Then the constants of integration in Equation (233) can be evaluated from these boundary conditions; the final result is

$$u = \left(\frac{dp}{dx}\right)\frac{1}{2\mu}(y^2 - hy) + \frac{V}{h}(h - y). \tag{234}$$

A rise in pressure in the positive direction of x corresponds to a positive value of dp/dx, and a fall of pressure corresponds to a negative value of dp/dx.

From the condition of continuity, the volume of incompressible fluid passing any cross section of the channel per unit time is

$$Q = \int_0^h u\,dy = \frac{Vh}{2} - \frac{h^3}{12\mu}\frac{dp}{dx}. \tag{235}$$

Equation (235) can be rewritten as

$$\frac{dp}{dx} = 12\mu\left[\frac{V}{2h^2} - \frac{Q}{h^3}\right]. \tag{236}$$

If h can be expressed as a function of x, the relation between p and can be found by integrating Equation (236). Since $h = (a - x)\delta$,

$$p = 12\mu\frac{V}{2}\int_0^x \frac{dx}{h^2} - 12\mu Q\int_0^x \frac{dx}{h^3}. \qquad (237$$

Integration of the separate terms gives

$$\int_0^x \frac{dx}{h^3} = \frac{1}{2\delta^3}\left[\frac{1}{(a-x)^2} - \frac{1}{a^2}\right] = \frac{2ax - x^2}{2\delta^3a^2(a-x)^2},$$

$$\int_0^x \frac{dx}{h^2} = \frac{1}{\delta^2}\left[\frac{1}{a-x} - \frac{1}{a}\right] = \frac{x}{a\delta^2(a-x)}.$$

Therefore, Equation (237) becomes

$$p = p_0 + \frac{6\mu x}{a\delta^2(a-x)}\left[V - \frac{Q(2a-x)}{a\delta(a-x)}\right], \qquad (238$$

where p_0 is the pressure existing at $x = 0$ (p_0 may be atmospheric pressure). The pressure p must also equal p_0 when $x = l$. This means that the term inside the bracket in Equation (238) must vanish when $x = l$, that is,

$$Q = \frac{Va\delta(a-l)}{2a-l}. \tag{239}$$

Substituting h for $\delta(a - x)$ and simplifying the result yields

$$p = p_0 + \frac{6\mu Vx(l-x)}{h^2(2a-l)}. \tag{240}$$

If the surfaces are parallel, a approaches infinity, and Equation (240) shows that the pressure p equals p_0 for *all* distances x in the bearing. Thus there is no pressure (above p_0) developed in the lubricant film. If a has a finite value, if the surfaces are *not* parallel, then the second term of Equation (240) has a finite value, and the pressure in the film is above p_0.

Equation (240) can be expressed without the variable h as

$$p = p_0 + \frac{6\mu Vx(l-x)}{\delta^2(2a-l)(a-x)^2}. \tag{241}$$

Let the mean thickness of the film at $x = l/2$ be represented by h_m. Then

$$h_m = \frac{\delta}{2}(2a - l). \tag{242}$$

Substituting h_m in Equation (241) yields

$$p - p_0 = \frac{3\mu Vx(l - x)}{\delta h_m(a - x)^2}. \tag{243}$$

Equation (243) shows that even when the dynamic viscosity is relatively small, very high film pressures can be developed if the mean thickness, h_m, of the oil film is very small.

Equation (243) can be expressed in the dimensionless form

$$\frac{p - p_0}{3\mu \frac{V}{h_m}} = \frac{x(l - x)}{\delta(a - x)^2}. \tag{244}$$

For constant values of μ, V, and h_m, the variation in the left-hand term of Equation (244) gives an indication of the variation of the excess

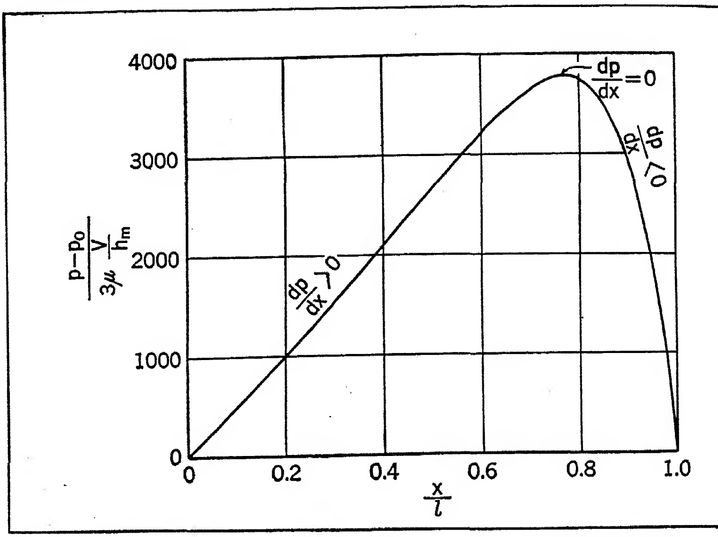


FIG. 185. Theoretical variation of pressure along the length of a slipper bearing. $l/a = 0.7$, h_m is 0.0012 inch, $l = 12$ inches.

pressure developed in the film. As an example, Fig. 185 shows the pressure distribution for a bearing in which $h_m = 0.0012$ inches, $l = 12$ inches and $l/a = 0.7$. Figure 185 shows that the pressure rises from p_0 to a maximum, and then drops to p_0. The maximum pressure is not at the center of the bearing. The maximum pressure, and its location, can be determined by differentiating p with respect to x and setting the result $dp/dx = 0$. In practice, thrust slippers are pivoted somewhat behind the middle in order to provide stable operation.

139. Other aspects of lubrication

This chapter has discussed very briefly only a few of the many phases of lubrication. Other phases of the subject are important. Further and more specialized information can be found in the literature. For example, there is the general problem of thermal equilibrium in a bearing. Bearing performance relations involve the viscosity of the lubricant under steady operating conditions. This viscosity can be determined if the operating temperature is known. Making a heat balance for a bearing and determining this temperature for a bearing before it is built is sometimes a difficult problem. The solution to this problem depends both on the rate of generating heat by friction and on the rate of heat transfer from the bearing to its surroundings.

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PROBLEMS

143. The ratio zN/p is frequently used in lubrication work, in which z is in centipoises, N is in revolutions per minute, and p is in pounds per square inch. zN/p, although involving a hybrid system of units, is convenient in correlating data. Determine the factor by which zN/p must be multiplied in order to get the ratio $\mu N/p$ in any set of consistent units.

144. By means of velocity profiles, explain why the maximum film pressure in Fig. 169 is nearer to the smaller opening at A than to B.

145. An oil has a Saybolt reading of 1500 seconds at 100° Fahrenheit and 90 seconds at 210° Fahrenheit. What is the viscosity index of the oil?

146. What is the dynamic viscosity in American units of a high viscosity index S.A.E. 30 motor oil at 70° Fahrenheit, and at 210 degrees Fahrenheit?

147. What is the kinematic viscosity in American units of a low-viscosity-index S.A.E. 10 motor oil at 0° Fahrenheit, and 210° Fahrenheit?

148. What is the A.P.I. gravity at 60° Fahrenheit of high-viscosity-index S.A.E. 60 aviation engine oil?

149. A shaft 3 inches in diameter rotates at 1800 revolutions per minute in a bearing 3 inches long with a diametral clearance of 0.003 inch. The total load on the journal is 900 pounds, and the lubricant is a light steam-turbine oil at 100° Fahrenheit. What is the power loss in the bearing?

150. If the load in Problem 149 were increased to 9000 pounds, while keeping everything else the same, what would be the theoretical minimum oil film thickness?

151. A bearing test gave a friction coefficient of 0.0122, a projected bearing pressure of 1.74 pounds per square inch, and a shaft speed of 93.4 revolutions per minute. The viscosity of the lubricant was 3.22 centipoises. If the Petroff law is assumed to apply, what was the ratio of diameter to clearance?

152. A journal 2 inches in diameter in a bearing 2 inches long rotates at 200 revolutions per minute with a total load of 1600 pounds. Select an oil such that the $\mu N/p$ ratio is not less than 0.3×10^{-6}, if the oil temperature is 130° Fahrenheit.

153. A shaft 4 inches in diameter in a bearing 5 inches long has a total load of 4000 pounds. The lubricant is a high-viscosity-index S.A.E. 30 motor oil, at an operating temperature of 130° Fahrenheit. Tests show that this type of bearing should not operate steadily below a $\mu N/p$ value of 0.4×10^{-6}. What is the lowest steady speed of the shaft for these conditions?

154. Starting with Equation (243), determine the value of x and the value of p at which the oil film pressure is a maximum.

155. In a slipper bearing $l = 12$ inches, $h_m = 0.0012$ inches, and $l/a = 0.7$. Determine the dimensionless value of x/l and the dimensionless value of $\dfrac{p}{3\mu \dfrac{V}{h_m}}$ for maximum oil-film pressure.

CHAPTER 16

Pumps

The present chapter and the next two might be classified under the general heading of fluid machinery. The purpose of these three chapters is to present briefly some general features and performance characteristics of a few common fluid machines. Pumps are treated first. Turbines are then considered in Chapter 17. Following this section, in the same chapter, is a discussion of fluid couplings and torque converters, each of which is essentially a combination of a centrifugal pump and a turbine. Chapter 18 reviews some basic fluid power and control systems.

140. Reciprocating pumps and motors

In the familiar reciprocating pump a piston moves back and forth in a cylinder. Figure 186 shows a diagrammatic sketch of one type. Fluid is forced through the discharge valve as power is applied to the piston. The reciprocating pump is usually classified as a positive-displacement machine. If leakage or slip is neglected, then the pump discharges a volume of fluid equal to the volume displaced by the piston. The discharge pressure of a reciprocating pump is governed by the discharge piping, or load, and not by the pump. For a given or constant volumetric rate of flow, the discharge pressure may be low or high, depending on the pressure necessary to force the flow through the system connected to the pump. The pressure that can be developed is limited by the strength of the pump and the power of the driving unit.

On many reciprocating pumps there is an air chamber on the discharge, to make the flow more steady and to make the pump operation quiet by cushioning the discharge. The air in the chamber is compressed during discharge. When the piston reaches the end of the stroke, expansion of this air tends to keep the fluid in motion until the reverse stroke starts.

In general, reciprocating pumps are most efficient for relatively small rates of discharge or capacities, high pressures, and high suction lifts. They are built for practically every type of service, and with a wide variety of materials to meet these services. Generally speaking, reciprocating pumps are not particularly well suited for handling very viscous or dirty liquids, because of the tendency toward clogging of suction and discharge valves. Reciprocating pumps are usually operated

at slow speeds (40 to 200 crankshaft revolutions per minute) because of the reciprocating motion and the valves.

The action of a reciprocating pump can be reversed to give a reciprocating *fluid motor.* A combination of pump, fluid motor, and interconnecting piping or channels may be called a fluid power transmission. A displacement-type fluid power transmission has one positive-displacement pump in which mechanical work is converted into liquid

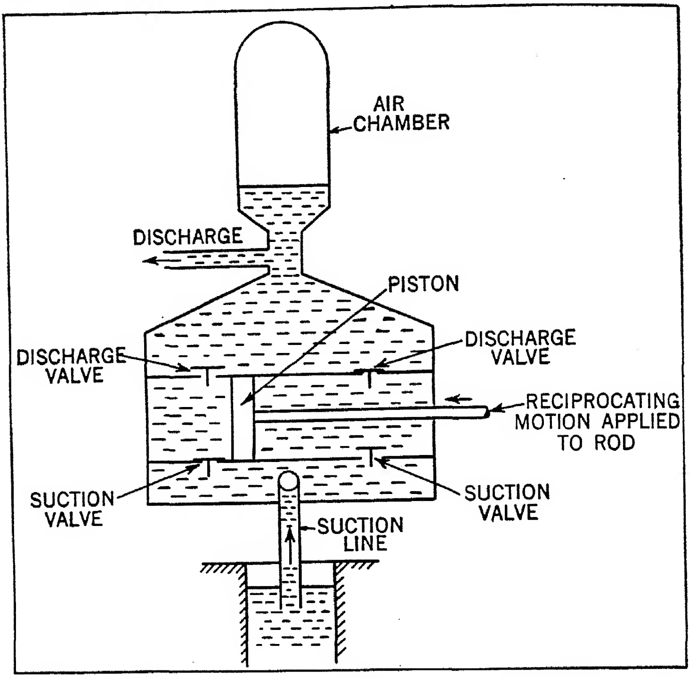


FIG. 186. Diagrammatic sketch of a reciprocating pump.

flow against pressure, and a second positive-displacement pump acting in reverse, as a motor, in which the liquid flow is transformed into mechanical work. The displacement-type power transmission differs from the so-called *hydrodynamic* transmission. The fluid coupling is an example of a hydrodynamic transmission. In the fluid coupling the kinetic energy of the liquid, rather than pressure alone, is used to transmit power.

141. Some other positive-displacement machines

The action in a rotary pump is one of rotation and not reciprocation. The rotary pump is commonly classified as a positive-displacement

machine. The flow from a reciprocating pump is pulsating, whereas the flow from many types of rotary pumps is fairly steady. A rotary pump is *not* a centrifugal pump. The rotary pump shown in Fig. 187 includes a pair of meshed gears in a casing. As the gears rotate, the fluid is trapped between the gear teeth and the case, and is carried around to the discharge. During each revolution of the gears a certain volume of fluid is transferred from suction to discharge, depending upon the size of the spaces between the gear teeth and the case. The pressure developed by a rotary pump is, as with a reciprocating pump, whatever is required to force the fluid through the discharge piping.

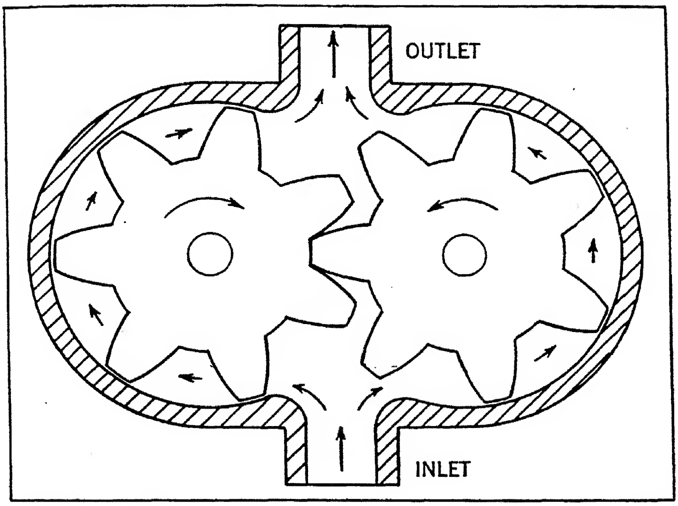


FIG. 187. Essential features of one type of rotary pump.

Rotary pumps are suitable for certain classes of service under low and medium head. The absence of valves is an advantage in the handling of thick, viscous liquids. Rotary pumps are not well suited for the handling of grit or abrasives because of the close clearances between gears and case. The action of the device shown in Fig. 187 can be reversed, to result in a fluid "motor."

The vane pump, as illustrated in Fig. 188, is another type of positive-displacement machine. The rotating member, with its sliding vanes, is set off center in the casing. The entering fluid is trapped between the vanes (which ride on the inside of the case) and is carried to the outlet.

Different designs of lobe-type rotary pumps have been devised. Figure 189 shows a two-lobe type. The two lobes, mounted on parallel shafts, rotate in opposite directions. A pair of timing gears, located at one end of the shafts, maintains the proper relation between the lobes

throughout rotation. Fluid is drawn into the space between the lobe and case, and pushed from inlet to outlet. Such units are used as blowers, for moving air or gas, or in reverse as meters for measuring gas flows. The rotary volumetric or displacement meter was discussed in Article 70. A supercharger on an aircraft engine is a pump for supplying the engine with a greater weight of air or fuel-air mixture than would normally be inducted at the prevailing atmospheric pressure. When the type

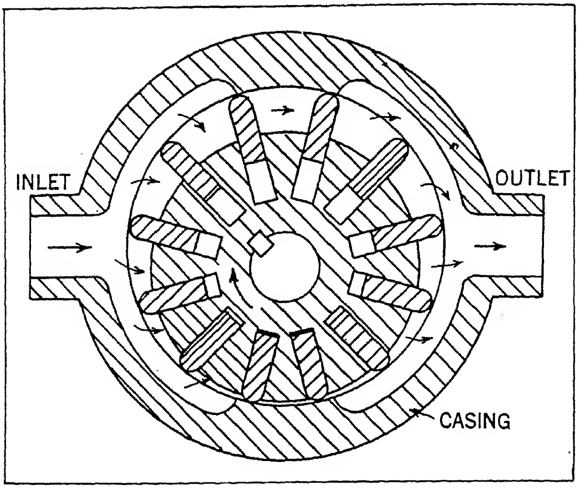


FIG. 188. Vane pump.

of unit shown in Fig. 189 is so used, it is frequently called a *Roots-type* supercharger.

142. Centrifugal pumps

Centrifugal pumps are one of the most popular types, because of their simplicity, compactness, low cost, and ability to operate under a wide variety of conditions. The action in a centrifugal pump depends partly upon centrifugal force or the variation of pressure due to rotation. This principle explains the particular name given to such a pump.

The essential parts of a centrifugal pump are a rotating member with vanes, or so-called *impeller*, and a case surrounding it. The impeller may be driven by a high-speed electric motor, an internal-combustion engine, or a steam turbine. Belt drives and direct connections are used. As indicated in Fig. 190, fluid is led through the inlet pipe to the center or *eye* of the rotating impeller. The rotating impeller throws the fluid into the *volute*, where it is led through the discharge *nozzle* to the discharge piping. The fluid leaves the impeller with a high velocity. An impor-

tant function of the pump passageways is to develop available pressure by an efficient conversion of kinetic energy.

The centrifugal pump differs from the reciprocating pump in many respects. The discharge valve of a centrifugal pump can be closed completely without causing the pressure to rise above a certain value. If the discharge valve is closed, the rotating impeller simply churns and heats the fluid. If the discharge valve of a reciprocating pump were

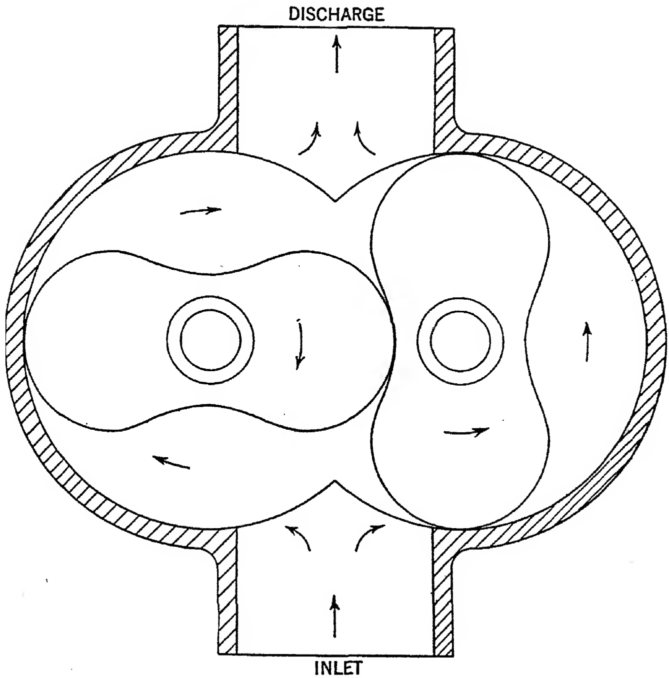


Fig. 189. Two-lobe rotary pump.

closed, the pump would be stopped or something would burst. The discharge from a centrifugal pump is relatively smooth and steady. Centrifugal pumps are available that can handle mashes, sewage, and liquids containing sand, gravel, and rocks of moderate size.

Centrifugal pumps are sometimes designated as: (a) volute-type pumps, or (b) diffuser-type pumps. Figure 190 shows a volute-type pump; the fluid is discharged directly from the impeller into the volute. In the diffuser-type (Fig. 191) there is a diffuser, consisting of a series of fixed guide vanes, surrounding the impeller. The function of the diffuser is to guide the fluid and reduce its velocity; there is a reduction

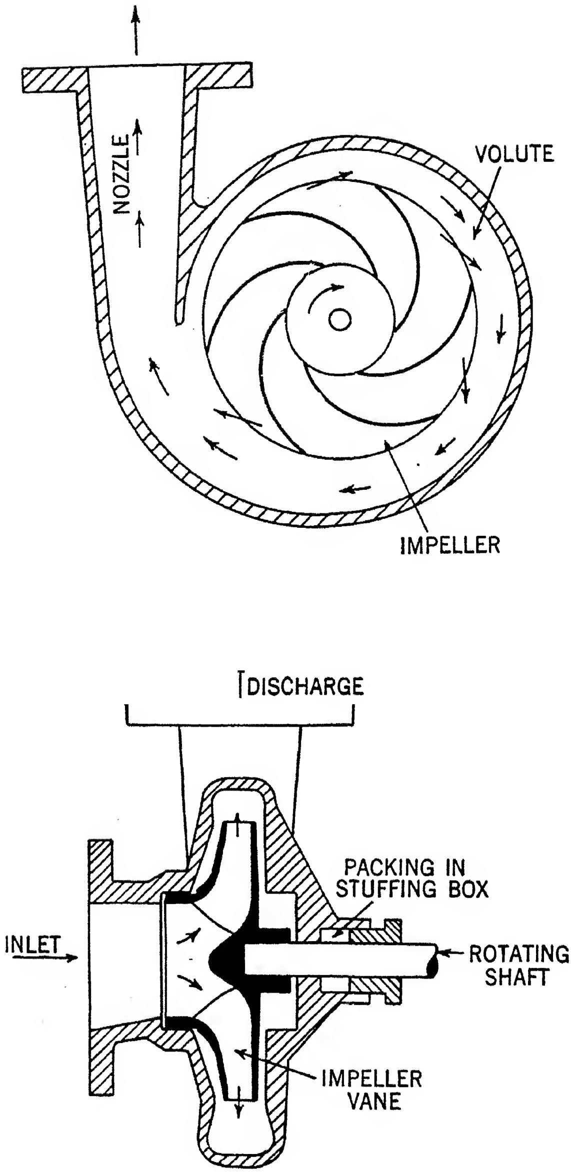


Fig. 190. Features of a volute-type centrifugal pump.

in kinetic energy and an increase in static pressure in the diffuser. The diffuser tends to make the static pressure distribution around the impeller uniform. Sometimes the diffuser-type pump is called a "turbine" pump, probably because its construction is similar to that of turbines having guide vanes. In commercial practice, however, the term "turbine" pump[1] is also applied to a pump differing in construction from the diffuser pump shown in Fig. 191.

Centrifugal pumps are built with many different arrangements of impellers, and with many other variations in details of construction. In

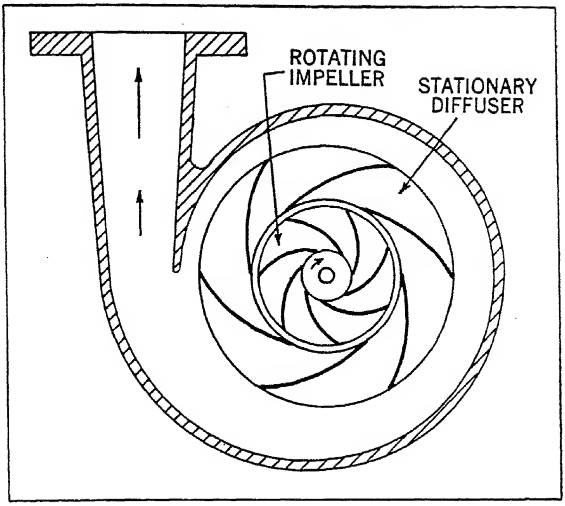
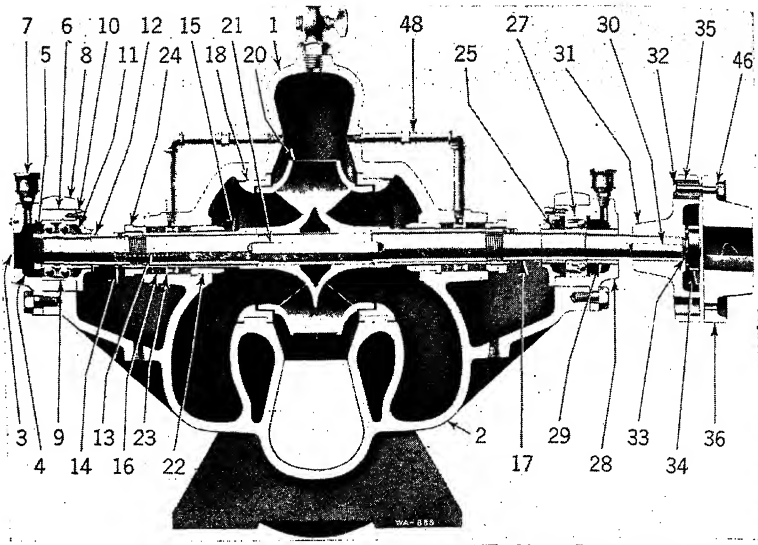


FIG. 191. Diffuser-type centrifugal pump.

the so-called *single-suction* pump, fluid enters the impeller eye from only one side of the impeller. Figure 190 shows a single-suction pump. In the so-called *double-suction* pump, fluid enters from both sides of the impeller. A pump may be *staged* with several impellers on one shaft, being then essentially several pumps in series. In a two-stage pump, for example, two impellers can be mounted on the same shaft in one casing. The discharge from the first impeller enters the inlet of the second impeller. The same weight of fluid per unit time flows through each stage, but each stage increases the pressure. Figure 192 shows a section through a commercial double-suction single-stage pump, together with a list of parts.

[1] A discussion of turbine pumps can be found in *Pumps* by Kristal and Annett. McGraw-Hill, New York, 1940, page 116.



Courtesy of Worthington Pump & Machinery Corp.

FIG. 192. Worthington double-suction single-stage centrifugal pump. (1) Upper half of casing; (2) lower half of casing; (3) bearing cap; (4) bearing lock nut; (5) bearing lock-nut washer; (6) thrust-bearing housing; (7) grease cup; (8) bearing housing strap; (9) thrust bearing; (10) thrust-bearing cover; (11) thrust-bearing spacer; (12) water shield; (13) gland nut and bolt; (14) shaft-sleeve nut; (15) shaft sleeve; (16) packing; (17) shaft; (18) impeller guide ring; (20) impeller; (21) impeller key; (22) stuffing-box bushing; (23) water-seal cage; (24) gland; (25) line-bearing spacer; (27) line bearing; (28) line-bearing housing; (29) line-bearing collar; (30) coupling key; (31) driven half coupling; (32) coupling pin and nut; (33) coupling-nut lock washer; (34) coupling nut; (35) rubber bushing; (36) driving half coupling; (46) coupling-pin lock washer; (48) water-seal piping.

143. Definition of performance terms

A careful examination of terms will eliminate confusion in studying pump performance. The term *head,* with various qualifications, is used in pump practice. Head of the fluid flowing is expressed in length units (such as feet) and equals energy per unit weight of fluid (such as foot-pounds per pound of fluid). The vertical distance from the surface of the supply source to the centerline of the pump shaft is called a *static* head. In Fig. 193, the distance from the pump shaft center line A to the level in the suction tank is variously called *negative static inlet head, negative static suction head,* or simply *static suction lift.* If the supply level is above the pump shaft center line, the static inlet head is positive.

The difference in *total* energy between the discharge and inlet openings of the pump, which is the energy added to the stream, represents the energy added by the pump. The term *total head,* or *total dynamic head,*

is employed to designate the energy added by the pump to unit weight of the flowing stream. Figure 194 shows the notation; the subscript 1 refers to inlet, and the subscript 2 refers to discharge. For steady incompressible flow, the total head H is

$$H = \frac{p_2 - p_1}{w} + \frac{V_2^2 - V_1^2}{2g} + (z_2 - z_1). \tag{245}$$

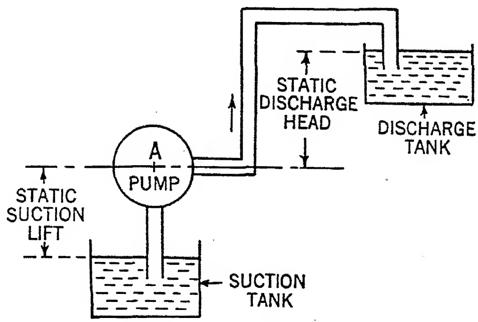


FIG. 193. Static heads.

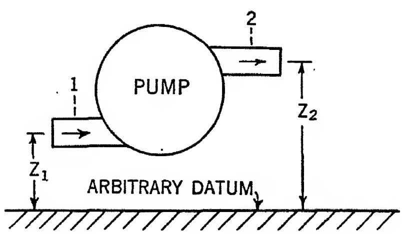


FIG. 194. Notation for pump.

The average velocity at exit V_2 is different from the velocity at inlet V_1 if the pipe diameters are different. The inlet pressure p_1 may be higher or lower than atmospheric. Calculating H should not cause any difficulty if it is borne in mind that $p_2 - p_1$ is a pressure *difference*.

The power added to the fluid by the pump is WH, where W is the weight flowing per unit time. For example, if the total head for a centrifugal pump handling an oil is 300 feet, the pump adds 300 foot-pounds of energy to each pound of oil flowing. Multiplying 300 by the number of pounds of oil flowing per second gives the power added to the oil in foot-pounds per second. The efficiency of a pump is equal to the power increase furnished to the fluid by the pump divided by the power input measured at the pump shaft. Pump input horsepower is sometimes called *brake horsepower.*

The *capacity* or *discharge* of a pump refers to the volume of fluid handled per unit time. Capacity may be expressed in such units as gallons per minute, or cubic feet per second. *Normal* or *rated* capacity usually represents the capacity at maximum pump efficiency.

144. Performance characteristics of centrifugal pumps

Figure 195 shows the performance characteristics of a modern centrifugal pump tested at constant shaft speed (the fluid is water). The

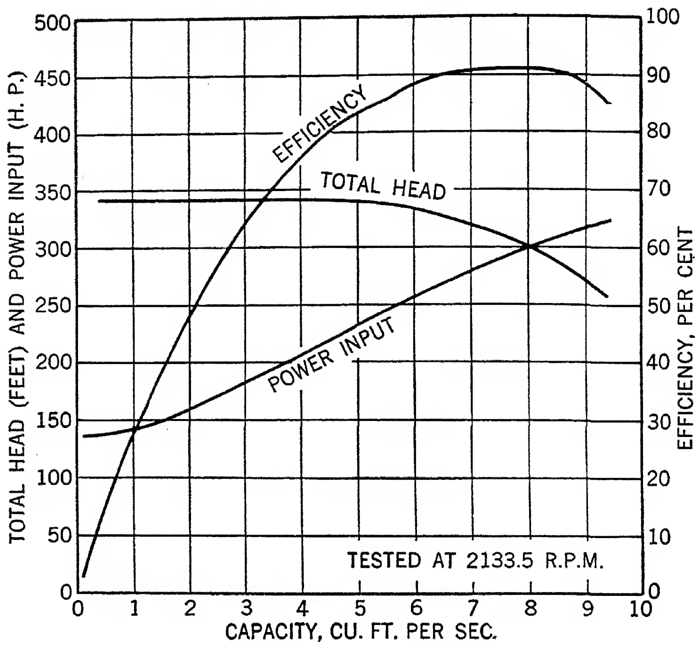


Fig. 195. Performance characteristics of a single-stage single-suction Byron Jackson centrifugal pump. Test made at California Institute of Technology for the Metropolitan Water District of Southern California.

efficiency rises to a peak value of 91.7 per cent and then drops. As illustrated in Fig. 196, a centrifugal pump may have a *rising*, *flat*, or *falling* characteristic for the relation between total head and capacity, depending upon the design. Figure 197 illustrates the general features of the pump characteristics obtained with different constant-speed tests. The normal capacity increases as the speed is increased.

145. Dynamic similarity relations

Much attention has been devoted to the general problem of developing analytical relations for predicting centrifugal-pump behavior. This problem is exceedingly difficult, because the nature of the complicated, turbulent flow inside a centrifugal pump has not been completely established at the present time. There are general, approximate relations, however, which are helpful in indicating relative performance trends. Some of these relations, based on studies of dynamic similarity, will be presented in the following paragraphs.

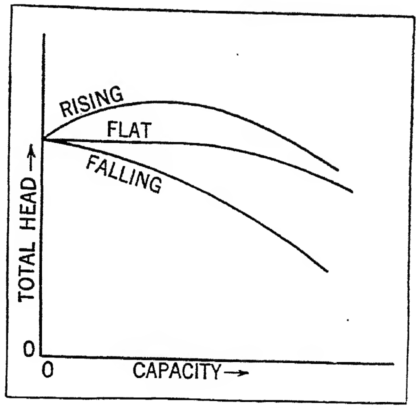


FIG. 196. Types of centrifugal-pump characteristics.

Figure 198a shows the velocity triangle at the exit from an impeller vane. The peripheral velocity u is proportional to the diameter of the impeller, D, and the revolutions per unit time, N, of the

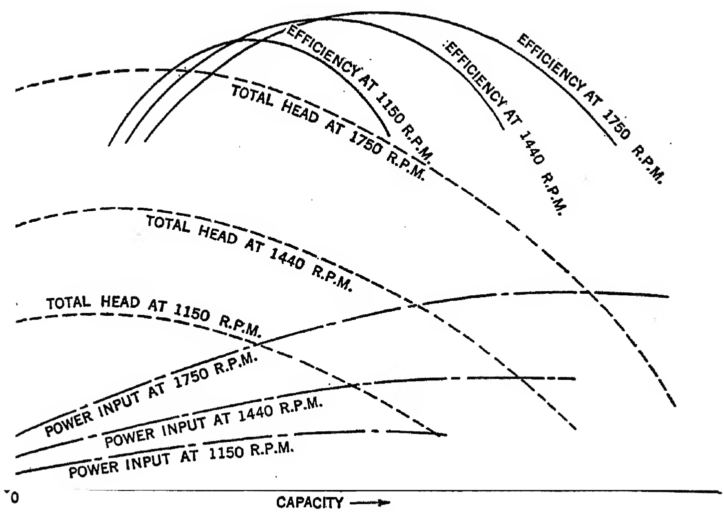
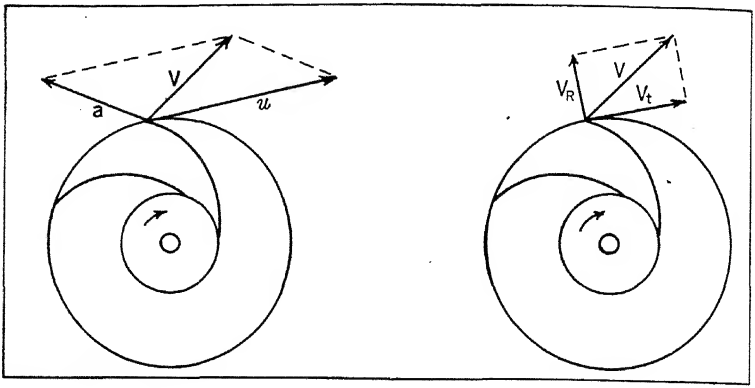


FIG 197. Performance characteristics of a centrifugal pump at different constant

impeller. The velocity a is the velocity of the fluid relative to the impeller. The absolute velocity of the fluid V is the vector sum of a and u. The radial component of V is V_R, as indicated in Fig. 198b.

Consider first a single centrifugal pump operating at different shaft speeds, at substantially the same relatively high efficiency. If the flow conditions are similar, the velocity triangles at the different speeds are similar. The capacity Q of a pump is proportional to V_R and the exit area of the impeller. V_R is directly proportional to N. For example, if N is doubled, u is doubled, V is doubled, and V_R is doubled. Thus, Q is directly proportional to N.



(a) (b)

FIG. 198. Velocity triangle at exit from an impeller vane.

Equation (88), Chapter 9, shows that the torque T acting on a rotating channel (such as the vane passage) is given by the expression

$$T = \frac{Qw}{a} (V_{t_1}r_1 - V_{t_2}r_2),$$

where V_t is the tangential component of the absolute velocity, r is the radius, and the subscripts 1 and 2 refer to conditions at two different radii (as at inlet to and exit from the vane passage). The work done by a torque equals the product of torque and the angle (in radians) through which it works. Since $2\pi N$ equals the number of radians per unit time, the work done by the vane passage per unit weight of fluid is

$$\text{work per unit weight} = \frac{T2\pi N}{Qw} \quad \frac{2\pi N}{g} [V_{t_1}r_1 - \qquad (246)$$

Equation (246) shows that the work per unit weight is directly proportional to the product of two linear velocities; namely, a tangential

component of the absolute velocity, and a peripheral velocity. For the mechanical energy exchange between the blades and the flow, the total head H is thus directly proportional to $(ND)^2$. A more familiar form, with the same dimensions on both sides, can be obtained by writing

$$H \text{ is directly proportional to } \frac{(ND)^2}{2g}. \tag{247}$$

The power added to the fluid by the pump is proportional to the product of Q and H.

Within the normal operation of one centrifugal pump at different speeds with similar flow patterns, the foregoing relations can be summarized:

Pump capacity varies directly as the speed N.
Total head varies directly as N^2.
Fluid power developed by the pump varies directly as N^3.

Consider next a series of geometrically similar centrifugal pumps of different sizes, but giving similar flow patterns, and operating at the same shaft speed. Equation (247) shows that the total head varies directly as the square of the impeller diameter. The peripheral velocity u is proportional to the product (ND). Since Q is proportional to the product of V_R times the exit impeller area, and the impeller area is proportional to D^2, then Q is proportional to $(V_R D^2)$. For similar flow patterns, the ratio V_R/u for one pump equals the ratio V_R/u for each of the other pumps; in other words,

$$\frac{V_R}{u} \text{ or } \frac{Q}{ND^3} \quad \text{a constant.} \tag{248}$$

The fluid power is proportional to the product of head and capacity.

For a series of geometrically similar pumps operating at one shaft speed, with similar flow patterns, the foregoing relations can be summarized:

Total head varies directly as D^2.
Capacity varies directly as D^3.
Fluid power developed by pump varies as D^5.

146. Specific speed of pumps

A dimensional analysis of the performance of a centrifugal pump yields, besides other ratios, the one significant dimensionless ratio

$$\frac{NQ^{1/2}}{(gH)^{3/4}}.$$

A factor similar to this ratio, only without g and with a particular set of units, is widely used in industrial practice. This factor is called *specific speed.* The specific speed of a pump, N_S, is defined as

$$N \frac{\sqrt{\text{G.P.M.}}}{H^{3/4}},$$

where N is the revolutions per minute of the pump shaft, G.P.M. is the capacity in gallons per minute, and H is the total head per stage in feet. In practice, the total head and capacity values used are those at the point of maximum efficiency for the pump shaft speed employed.

Specific speed is useful as an index of the type of pump; specific speed provides a parameter which is useful in determining what combinations of head, speed, and capacity are possible and desirable. Impellers for high total heads usually have low specific speeds, whereas impellers for low total heads usually have high specific speeds.

147. Propeller pumps

Pumps have been classified according to the general design of the vanes or blades on the rotating member and the shape of the casing. In the so-called *radial-flow* pump the fluid enters the impeller at the hub and flows radially to the periphery. Figure 190 is an example. The total head is developed principally by the action of centrifugal force. Pumps of this type with single-suction impellers usually have a specific speed less than 4200. A specific speed less than 500 is not considered practicable or desirable. Pumps of this class with double-suction impellers usually have a specific speed less than 6000.

In the so-called *mixed-flow* pump the fluid enters the impeller axially, and discharges in both an axial and radial direction, usually into a volute-type casing. The total head is developed partly by centrifugal force and partly by the dynamic lift of the vanes on the fluid. Pumps of this class usually have a specific speed between 4200 and 9000.

The propeller pump is variously called an axial-flow, straight-flow, or screw pump. The propeller pump has a single inlet impeller or rotating member with vanes or blades. A propeller pump might be visualized as a propeller in a circular fixed housing. The flow enters axially, and discharges nearly axially through a series of fixed guide vanes. The total head is developed principally by the propelling or dynamic lifting action between vanes and fluid. The action of lifting vanes and propellers was discussed in Chapter 11. Propeller pumps usually have a specific speed higher than 9000. Propeller pumps are used for such combinations as those with heads less than 40 feet and capacities greater than 300 gallons per minute, for drainage, irrigation, and condenser circulating-water installations. Figure 199 shows the general trend of

characteristics obtained with a propeller pump. Note that the head for a propeller pump drops at a greater rate with capacity than the head for an average centrifugal pump.

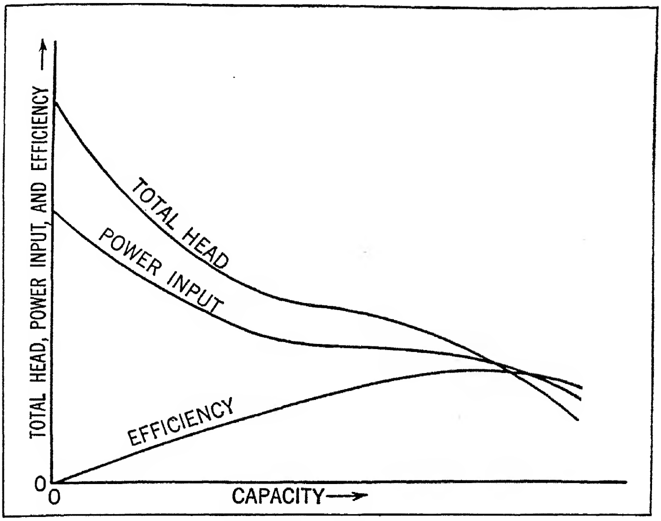


FIG. 199. Characteristics of a propeller pump at constant shaft speed.

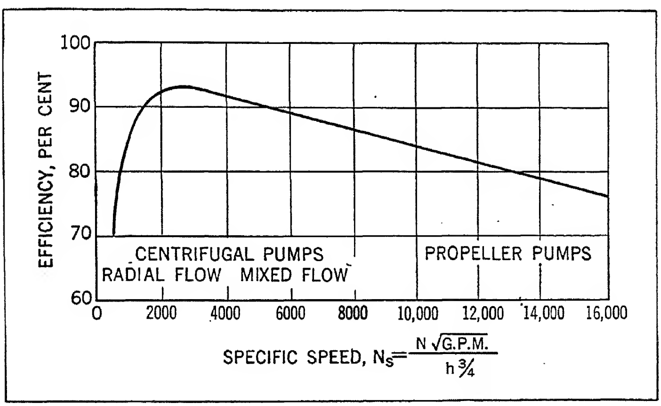


FIG. 200. Correlation between optimum efficiency and specific speed. (Curve supplied through courtesy of A. Hollander, Consulting Engineer, Byron Jackson Company.)

Figure 200 shows a correlation between optimum efficiency and specific speed for a range of pump types. The efficiency rises, reaches

a maximum, and then decreases. Unfavorable operating conditions and crude designs will give efficiencies lower than those shown in Fig. 200.

148. Pump systems

The pump is usually only one part of a system. The entire arrangement should be analyzed as a unit in order to obtain the most economical combination of equipment. Equation (245) defines the total head developed by a pump. The pump may be required to lift the fluid a certain vertical distance and to overcome system flow losses, such as those in the intake and discharge pipes. The calculation of pipe flow losses has been discussed in Chapters 7 and 13. Losses due to valves and fittings may be estimated by using the factors given in Table 6, Chapter 7. Minor losses in pipe lines, such as those caused by changes in sections, have been treated in Chapter 7.

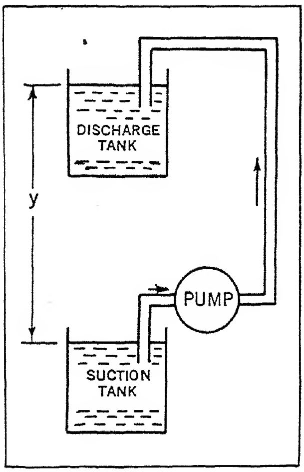


FIG. 201. Diagrammatic layout of a pumping system.

As an illustration, consider the system indicated in Fig. 201, with a certain static head y. The head-capacity characteristic for the pump at a certain speed is given in Fig. 202. The dotted horizontal line through point A represents the static head that must be overcome regardless of the flow. The head loss in pipe, fittings, and valves can be determined at various rates of flow. Addition of these system flow losses and the static head y at various capacities gives the service characteristic curve AC. The point of intersection B gives the head and capacity Q_1 at which the system will operate.

The system cannot have a capacity greater than Q_1. If it is desired to reduce the capacity below Q_1, some throttling of the flow, as with a valve, would be necessary. Service characteristics for different pipe diameters can be determined, and the corresponding operating capacities, heads, power inputs, and efficiencies found. A study of the effects with different pipe sizes is useful in determining the most economical pipe size.

For a liquid pump with a static suction lift, as shown in Fig. 201, the pressure at the pump inlet must be below atmospheric in order to cause flow from the suction tank to the pump. Theoretically speaking, the maximum suction lift is about 33.9 feet for water at 50° Fahrenheit. Actually, however, this maximum suction lift is never realized. There are pressure losses due to the flow. If the temperature of the liquid is high, the liquid may boil or vaporize at the low inlet pressure. Trying

to pump a vapor with a pump designed for liquids may cause difficulties. Practical limits of suction lifts for water at 60° Fahrenheit are frequently set at about 15 feet for some centrifugal pumps, and 22 feet for rotary and reciprocating pumps. The distances are less for higher temperatures.

Various parameters, like specific speed, are employed for establishing permissible suction lifts. A detailed discussion of these upper limits will not be attempted here. It should be borne in mind, however, that the suction conditions are among the most important factors affecting the operation of pumps handling liquids. An abnormally high suction

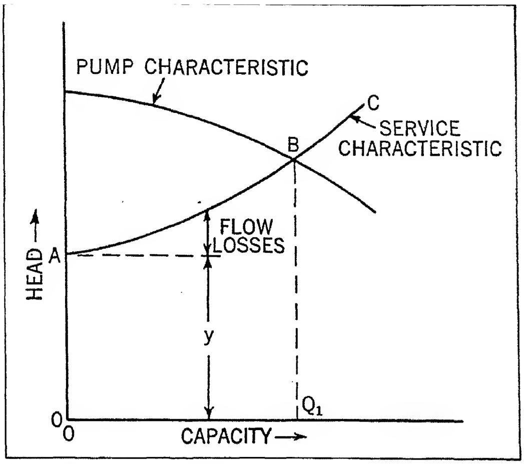


Fig. 202. Determination of the operating point of a pumping system.

lift usually causes serious reductions in capacity and pump efficiency, and may cause erosion or pitting of the pump parts. These effects will be discussed in Article 155 in the next chapter.

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PROBLEMS

156. A pump discharges 2000 gallons of brine (specific gravity = 1.2) per minute. The pump inlet, of 12-inch diameter, is at the same level as the outlet,

of 8-inch diameter. At inlet the vacuum is 6 inches of mercury. The center of the pressure gage connected to the pump discharge flange is 4 feet above the discharge-flange center. This gage reads 20 pounds per square inch gage. For a pump efficiency of 82 per cent, what is the power output of the motor?

157. A water pump develops a total head of 200 feet. The pump efficiency is 80 per cent and the motor efficiency is 87.5 per cent. If the power rate is 1.5 cents per kilowatt-hour, what is the power cost for pumping 1000 gallons? (1 horsepower = 0.746 kilowatts.)

158. A test on a centrifugal pump operating at 1150 revolutions per minute showed a total head of 37.6 feet at a capacity of 800 gallons per minute. Estimate the total head and capacity if this pump were operated at 1750 revolutions per minute. Assume point of maximum efficiency in each case.

159. A test on a centrifugal pump operating at 1150 revolutions per minute showed a total head of 40.0 feet at a capacity of 600 gallons per minute. The impeller diameter is 10.5 inches. Estimate the total head and capacity of a geometrically similar pump at 1150 revolutions per minute with an impeller diameter of 10.0 inches.

160. A centrifugal pump operating at 1800 revolutions per minute develops a total head of 200 feet at a capacity of 2500 gallons per minute. What is its specific speed?

161. Prove that the specific speed for the same impeller does not change with the impeller speed. (*Hint:* Express the head and capacity at one speed in terms of the head and capacity at another speed.)

162. For a certain system it is required to select a pump that will deliver 2400 gallons per minute at a total head of 360 feet, and a pump shaft speed of 2900 revolutions per minute. What type of pump would you suggest?

163. For a certain system it is required to select a pump that will deliver 2400 gallons per minute at a total head of 20 feet, and at a pump shaft speed of 2600 revolutions per minute. What type of pump would you suggest?

164. It is desired to pump 1000 gallons of gasoline (specific gravity = 0.85) per hour from the bottom of one tank to the top of another. The level in the inlet tank is 4 feet above the pump center line, and the level in the discharge tank is 96 feet above the pump center line. The inlet-pipe diameter equals the discharge-pipe diameter of 2 inches. The dynamic viscosity of the gasoline is 0.80 centipoises. The total length of clean steel pipe is 125 feet. If the pump efficiency is 80 per cent, what power output of the motor is required?

165. In a pumping system handling water (at 59° Fahrenheit) the level in the suction tank is 10 feet below the pump-shaft center line, and the level in the discharge tank is 70 feet above the pump-shaft center line. The inlet piping is 3 inches in diameter and together with its valves and fittings is equivalent to 84 feet of straight, clean steel pipe. The discharge line, $2\frac{1}{2}$ inches in diameter, with its valves and fittings is equivalent to 235 feet of straight, clean steel pipe. If the motor delivers 9.5 horsepower to the pump shaft, what is the pump efficiency for a flow of 200 gallons per minute?

166. Consider that the following quantities are involved in the performance of a centrifugal pump: capacity Q, shaft speed N, impeller diameter D, fluid density ρ, dynamic viscosity μ, and the quantity $E = gH$, where H is total head. E is energy per unit mass. By dimensional analysis, determine the form of the physical equation involving these variables.

CHAPTER 17

Turbines, Fluid Couplings, and Fluid Torque Converters

149. Types of turbines

In a pump, mechanical work (at a rotating shaft or a reciprocating rod) is transformed into fluid energy. A turbine is essentially a machine for transforming fluid energy into mechanical work at some rotating shaft. Except for the direction of the energy conversion, centrifugal

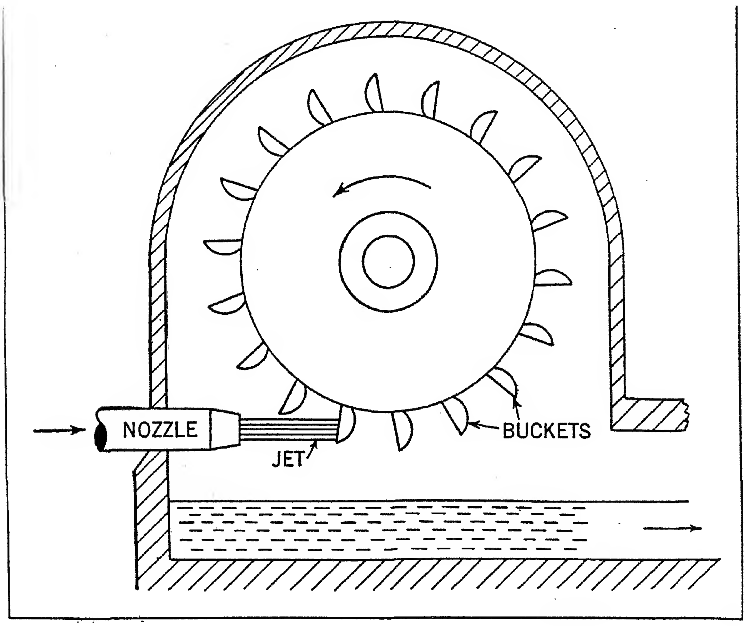


FIG. 203. Impulse turbine.

pumps and one common class of turbines (reaction) have many features in common. The efficiency of each type of machine depends on the shape of the blades. In principle it is possible to use the same blading for either. Knapp,[1] for example, reported results of experiments on a

[1] *Complete Characteristics of Centrifugal Pumps and Their Use in the Prediction of Transient Behavior* by R. T. Knapp, *A.S.M.E. Transactions*, November, 1937, page 683.

high-head, high-efficiency centrifugal pump. This machine was operated both as a turbine and as a pump. High efficiencies of the same order of magnitude were found in both the normal pump and turbine regions of operation.

Turbines are commonly divided into two general types: (a) impulse, and (b) reaction. In both types the momentum of a stream of fluid is changed by passing it across some sort of wheel, or runner, with buckets or vanes. In both types the force acting on the buckets or vanes rotates the runner, which performs useful work, and the fluid falls away with reduced energy. The terms *impulse* and *reaction* alone do not provide a complete distinction between the action in the two types of machines. The presence or absence of a free jet is one feature which distinguishes between the two types.

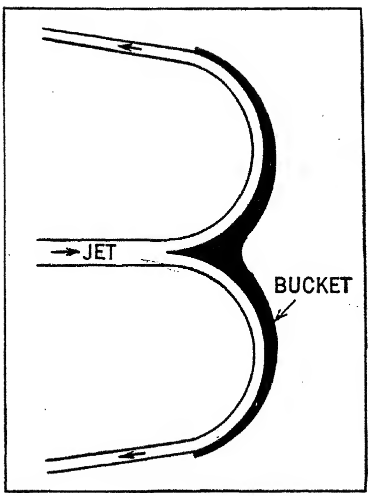


Fig. 204. Pelton-wheel bucket.

150. Impulse turbines

In turbines of the impulse, or so-called Pelton type, the fluid energy, first in the potential form, is next converted wholly into the kinetic form by means of a free jet in one or two nozzles. In the jet the static pressure is practically that of the atmosphere in which the jet is moving. As indicated in Fig. 203, this circular free jet strikes buckets on a rotating wheel. In practice these buckets (see Fig. 204) are usually spoon-shaped, with a central ridge splitting the impinging jet into two halves which are deflected backward through an angle of about 165°. A complete reversal of 180° would be desirable. This is not possible because the fluid must be thrown to one side in order to clear the following bucket. The actual energy transfer from jet to wheel is wholly by means of a reaction, that is, by changing the momentum of the stream.

The action on a single vane of an impulse turbine is indicated in Fig. 205. The subscript 1 refers to entrance and the subscript 2 refers to exit. Let u represent the peripheral velocity of the wheel, and a the relative velocity of the fluid with respect to the vane. The absolute velocity of the fluid V is the vector sum of a and u. The dotted curve in Fig. 205 represents the free edge of the jet at atmospheric pressure. The passage between the vanes is not completely filled with fluid. Under

favorable circumstances the exit velocity V_2 is nearly at right angles to the plane of rotation, and is very small. If the friction is very small, or neglected, then a_1 equals a_2 in magnitude. If the entrance point is at the same radius as the exit point, then u_1 equals u_2 and the flow is in an *axial* direction, a direction parallel to the axis of shaft rotation. In actual wheels the flow is approximately axial. The peripheral velocity of the wheel for maximum efficiency is slightly less than one-half the absolute velocity of the jet.

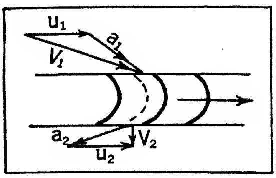


FIG. 205. Velocity diagram for an impulse-turbine wheel.

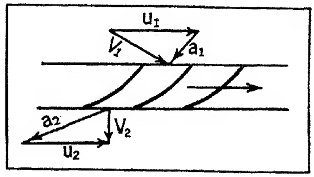


FIG. 206. Velocity diagram for a reaction-turbine runner.

151. Reaction turbines

There is no formation of a free jet in the reaction turbine. In this type there is a casing fitted with guide vanes completely surrounding the runner. The runner is a wheel provided with vanes, and fluid enters it completely around the periphery. In the reaction turbine only a moderate amount of the available fluid energy is transformed into kinetic energy before entrance to the runner, so that the fluid enters the runner with an excess pressure. The fluid completely fills the vane passages. Although the path of the fluid through the runner is tortuous, the essential features of the diagram in Fig. 206 apply. The relative velocity of the fluid is increased as it flows in the narrowing passages between the blades. Under favorable circumstances the absolute velocity after passing through the runner is nearly at right angles to the plane of rotation, and is very small.

Impulse turbines of modern design usually have horizontal shafts. Reaction turbines can be of either the vertical- or horizontal-shaft type. Probably because of economical reasons, the vertical unit seems to be more common in recent installations.

152. Turbine runners

In a manner which is somewhat similar to that for centrifugal and propeller pumps, turbines are sometimes classified as: (a) tangential (Pelton); (b) radial and mixed flow (Francis or American); and (c) axial flow (Kaplan). The Pelton wheel is called a *tangential* wheel because

the center line of the jet is tangent to the path of the centers of the buckets.

As represented in Fig. 207, fluid under pressure enters a spiral casing or housing which encompasses the runner. After flowing through

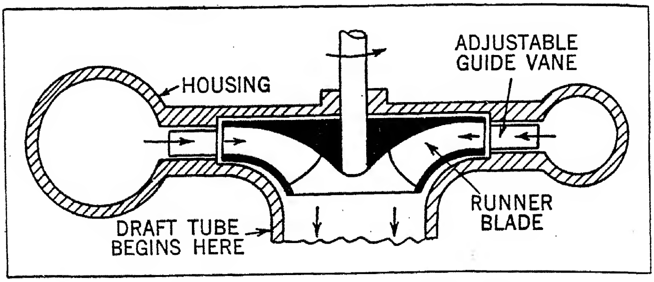


Fig. 207. Radial-flow turbine (Francis).

adjustable guide vanes, the fluid passes through the rotating runner in a plane practically normal to the axis of rotation; the flow is largely radially inward. This machine is frequently called a *radial-flow* or Francis turbine.

The flow is partly radial and partly axial in the runner shown in Fig. 208. This is known as *mixed flow*. This type of turbine is sometimes called the American type, although the name Francis is frequently extended to designate all inward-flow types of turbines. The Francis runner may have sixteen blades or vanes. A section of an actual installation is shown in Fig. 209. The turbine runner is connected directly to the vertical generator shaft.

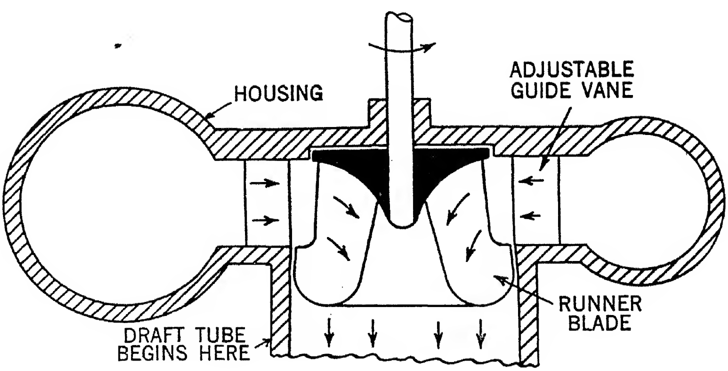
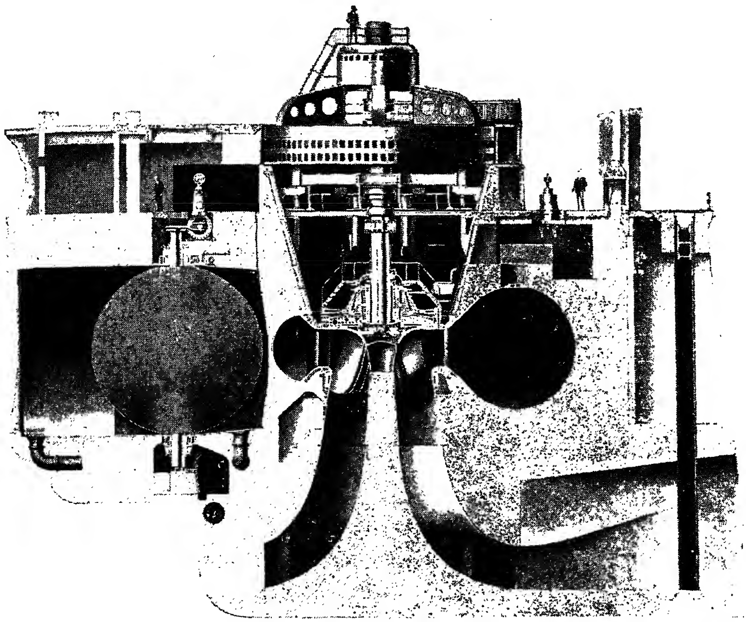


Fig. 208. Combined radial-axial-flow turbine.



Courtesy of the Baldwin Southwark Division, Baldwin Locomotive Works

FIG. 209. Sectional elevation of a 54,000-horsepower turbine, Conowingo Development, Susquehanna Power Company.

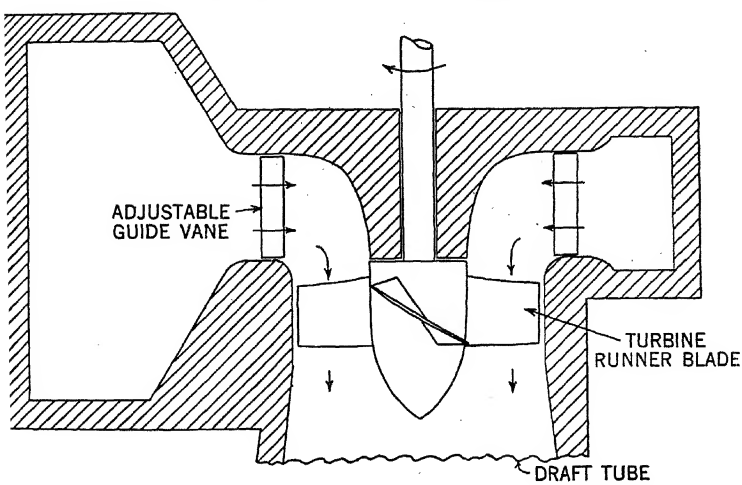


FIG. 210. Axial-flow turbine.

Turbines of the type shown in Fig. 210 are variously called *axial-flow* or *propeller type*. The arrangement of guide vanes is similar to that of the Francis turbine. The guide vanes give a spiral inward motion to the fluid. The fluid then passes axially through the runner, which may have four to six blades. If there is not much variation in available head and if the load is fairly constant, the *fixed-blade* propeller type of turbine runner is most economical. High efficiency can be maintained with a minimum number of mechanical parts. In some installations, however, the available head or the load may be variable. The Kaplan adjustable-blade propeller type of runner has been employed for such services. In this runner the pitch of each blade can be controlled to meet different conditions. The variable-pitch Kaplan runner is analogous, in some respects, to the variable-pitch propeller used on airplanes.

In many installations the amount of fluid entering the runner must be regulated, in order to adapt the turbine to suit the load at any time. This regulation is usually done automatically in such a way that the rotational speed of the turbine is kept constant. In the Pelton wheel, speed regulation is sometimes obtained by means of a regulating needle or spear in the jet. In reaction turbines, the adjustable guide vanes are rotated so that the channels are widened or narrowed.

153. Some performance features of water turbines

Generally speaking, the tangential or impulse wheel is used for high heads and relatively small volumetric rates of flow. The radial- and mixed-flow turbine is used for intermediate heads and intermediate rates of flow. The axial-flow turbine is used for low heads at high rotational speeds and large rates of flow.

Impulse turbines are generally employed for heads above 800 feet. A 3000-horsepower plant at Fully, Switzerland, has a head of 4850 feet. In this country the Southern California Edison Company has a plant with a head of 2400 feet. The efficiency of an impulse turbine may be more than 85 per cent; an average value is about 82 per cent.

Reaction turbines are generally regarded as not suited for heads greater than 800 feet, and are commonly employed under considerably lower heads. Intermediate heads between 15 and 750 feet have been used in economical installations. An average efficiency for reaction turbines might be placed at 90 per cent. Values slightly above this have been attained in large plants. The vertical-shaft 115,000-horsepower Boulder Dam reaction turbines have a rated head of 480 feet. The Kingsbury type of thrust bearing has made an important contribution to the development of heavy, large-size reaction turbine runners.

Units of the propeller type operate under heads up to 100 feet. Axial-flow turbines have been built with efficiencies above 90 per cent in units up to 60,000 horsepower. In the Bonneville project are 66,000-

horsepower Kaplan turbines designed to operate under heads varying from 30 to 69 feet.

The factor *specific speed* is used in turbine practice. The specific speed of a turbine, N_S, is defined as

$$N_S = \frac{N\sqrt{\text{B.HP.}}}{h^{5/4}} \tag{250}$$

where B.HP. is the brake horsepower, or power output at the rotating turbine shaft, h is the total available head, and N is the runner speed, in revolutions per minute, at which the maximum efficiency under a given head is attained. Specific speed is a characteristic index number; it serves to classify a turbine and to indicate its type. Considering average values, for impulse wheels N_S ranges from 0 to 4.5, for reaction turbines N_S varies from 10 to 100, whereas for propeller turbines N_S varies from 80 to 200.

154. Power plants

The turbine is but one part of a system. A turbine, with oil as the working fluid, is part of a fluid coupling, and a fluid torque converter. Steam and water turbines are used in power-generating plants. The choice as to whether a thermal or a water-power plant should be used depends upon a careful evaluation of many factors. Thermal and water plants are combined in some power systems. The present discussion is not intended to cover all the economic and social aspects involved in power generation, but only to illustrate very briefly some of the technical features of water-power plants.

A hydroelectric power plant consists mainly of means for delivering available water under pressure to a site, machinery and equipment to produce the power, the necessary turbine setting or foundation, and a building. A storage reservoir may be used in order to hold enough water to operate the plant for some length of time. In some plants there is no such reservoir. A dam of some sort may be used, to develop most or a portion of the available head. The intake equipment usually includes racks or screens to prevent trash from entering the runners, and a head gate to shut off the water flow if necessary.

The water is conducted toward the powerhouse by a conduit which may be an open channel, called a canal or flume, or a tunnel. In high-head plants, as a rule, water must be conducted considerable distances. A forebay is often placed at the end of the conduit. The forebay is a small reservoir to equalize the flow. The water is led from the forebay directly to the turbine through pressure pipes or penstocks. The turbine with its case or pit, and possibly a draft tube, comprise the setting. The turbine discharges into a body of water called the tail

water. The channel conducting the water away is called the tail race.

As indicated in Fig. 211, draft tubes are used in connection with reaction turbines. If a draft tube is employed, the entire available head can be used, and the turbine can still be set high enough above the tail-water level to allow inspection and maintenance without draining the tail bay. Practically, the draft head should not be greater than 12 to 15 feet, to prevent the water column from breaking. Descriptions and

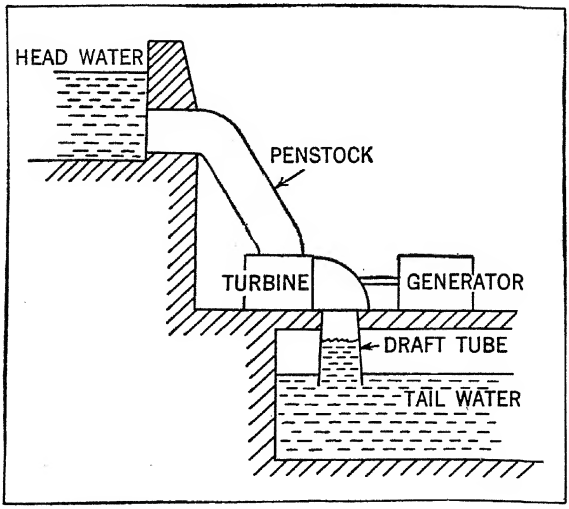


FIG. 211. Diagrammatic sketch of a water-power plant.

details of modern power plants can be found in such current references, as recent issues of *Mechanical Engineering* and the *A.S.M.E. Transactions.*

155. Cavitation

If a liquid flows through a machine or a stationary passage, there may be an unfavorable change in performance as operating conditions vary. There may be two serious undesirable effects: (1) a marked drop in efficiency, and (2) a dangerous erosion or pitting of some of the metal parts. The operation of a centrifugal pump handling water will be cited as an illustration. For a certain inlet head, if the discharge valve is opened, the efficiency rises to some peak as represented in Fig. 212. The sound emitted by the pump is normal. As the discharge valve is opened further, a condition may be reached at which the efficiency (and total head) drops markedly. This condition is sometimes called the *cutoff* point. As this condition is approached, the sound emitted by the pump changes. As first it sounds as if sand were passing through

the pump (with clear water entering). Then the sound or noise may change (as the discharge is increased) to give the impression of rocks passing through the pump, or a machine-gun barrage. If the pump is operated for any length of time at these conditions, the impeller may be badly eroded and pitted.

These two effects, of efficiency drop and pitting, have been discovered in the operation of water pumps, water turbines, marine propellers (on

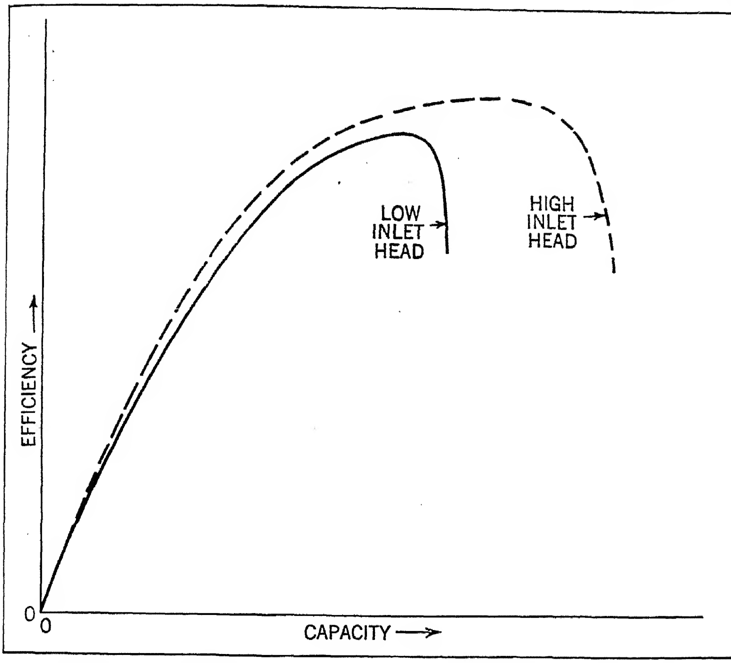


Fig. 212. Characteristics of a centrifugal pump at different inlet heads.

surface ships particularly), and in diverging channels, as in venturi tubes. In many cases these effects have been traced and ascribed to a phenomenon called *cavitation.* The word cavitation itself implies a cavity or a void. If, at some point in the liquid flow, the existing fluid pressure equals the vapor pressure at the particular temperature, then the liquid will vaporize—a cavity or void will form. If the fluid pressure fluctuates slightly (which fluctuation is common) above and below the vapor pressure, there will be an alternate formation and collapse of the vapor bubbles. Evidence shows that this alternate collapse and formation of bubbles is responsible for the marked drop in efficiency and the pitting of the metal parts.

Courtesy of the Baldwin Southwark Division, Baldwin Locomotive Works

FIG. 213. Hydraulic turbine runners showing the results of cavitation.

The violent collapse (taking place in a very short time) of vapor bubbles can force liquid at high velocity into the vapor-filled pores of the metal. The sudden stoppage at the bottom of the pore can produce surge pressures of high intensity on small areas. The process might be called an explosion or implosion. These surge pressures can exceed the tensile strength of the metal, and progressively blast out particles and give the metal a spongy appearance. Rapid pitting takes place, often eating holes through metal vanes, and dangerously weakening the structure. Figure 213 shows the results of cavitation on some hydraulic turbine runners; such evidence explains why some engineers are very much concerned about cavitation.

It is to be noted that two phases of a substance, liquid and vapor, are involved in the cavitation process. This double character makes the phenomenon difficult to analyze completely. Furthermore, in actual applications water has a small percentage of absorbed and dissolved air, oxygen, and nitrogen. These gases help the bubble formation. Solid matter may act as a catalyzer in the formation of bubbles. There is evidence to show that cavitation effects are purely mechanical. Some chemical action, however, may be present to supplement cavitation and to accelerate pitting.

Much theoretical and experimental work has been done on cavitation. The mechanism of cavitation has not been definitely or completely established yet on a quantitative basis for all fluid machines. Various parameters, like specific speed and dimensionless ratios including the vapor pressure or head, have been used to advantage in correlating experimental data on cavitation limits. It is difficult to calculate exactly the pressure in some of the complicated flows involved, as at a certain point on a pump impeller or a marine propeller. It is frequently necessary to investigate experimentally the performance of a machine, or its model, to determine upper limits below which cavitation effects do not occur. Sometimes special tests and laboratory arrangements are necessary, as for major water-pump and water-turbine installations.

In some cases, as for the diverging channel in a venturi tube, it may not be difficult to calculate the liquid pressure at various points. A comparison between the liquid pressure and the vapor pressure at the particular temperature should give some indication as to the possibility of cavitation occurring. Values of vapor pressure for water at different temperatures are given in Fig. 214. The variation of specific weight for water with temperature is given in Fig. 2.

For centrifugal pumps, particular care should be taken to have a high enough inlet pressure to avoid cavitation. In general, as the pump inlet pressure is reduced, the capacity at which cavitation occurs is reduced, as indicated in Fig. 212. It may be a greater calamity for a user to

secure a pump which is found to operate in the cavitating region than to lose 1 or 2 per cent in efficiency.

A clear distinction should be drawn between cavitation in liquids and the compressibility or shock-wave effect found in air or gases. These two phenomena are different. The shock-wave effect involves only one fluid. The formation of the shock wave depends upon whether or not

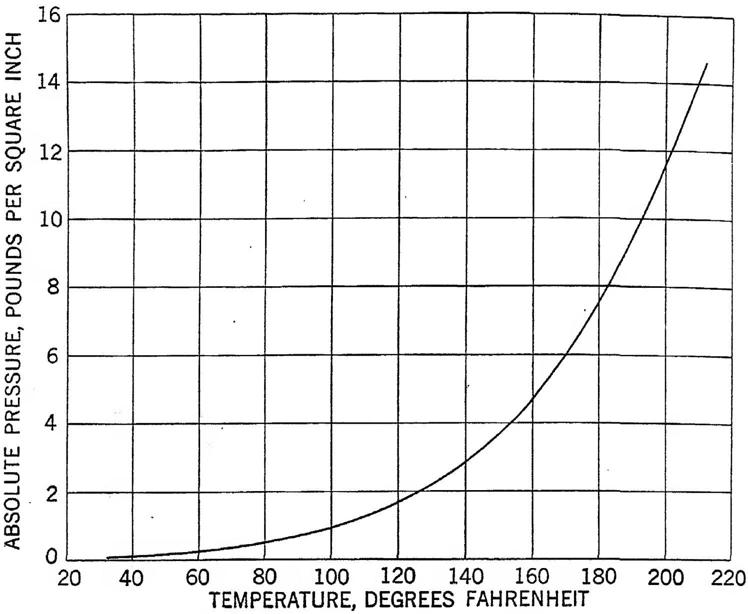


FIG. 214. Vapor pressure plotted against temperature for water. (Data adapted from *Thermodynamic Properties of Steam* by J. H. Keenan and F. G. Keyes. Wiley, New York, 1936.)

the fluid moves more slowly or faster than the velocity of pressure propagation. Cavitation involves two fluids.

156. Fluid couplings

Various methods can be employed for coupling or joining two rotating shafts, as for connecting an engine or motor to its load. Purely mechanical, rigid, or flexible couplings are used in many applications. In some applications, however, service requirements are best met by some form of fluid connector.

A fluid coupling is simply a combination of a centrifugal pump and a

turbine. Oil is usually employed in commercial units for the working fluid because of its lubricating properties, stability, and availability. Some recommendations call for a straight mineral lubricating oil having a maximum Saybolt reading of about 180 seconds at 130° Fahrenheit. The principle of the fluid coupling can be easily demonstrated by means of two ordinary electric fans which are set facing each other. One fan, connected to an electric outlet, is put into motion by turning on the electric current. As its blades rotate the air current which they develop turns the blades of the other fan which is not receiving any electric current.

As indicated in Fig. 215, the rotating input, or primary, shaft A drives the pump impeller, which usually has straight radial vanes. Kinetic energy is added to the fluid as the pump builds up speed, and the fluid flows outward. After sufficient energy has been developed, the fluid rotates the turbine runner and output, or secondary, shaft B. The fluid moves through a closed path shaped like a vortex ring. There is *no* mechanical or rigid connection between shafts A and B. The connection is solely by means of a fluid, which has cushioning properties.

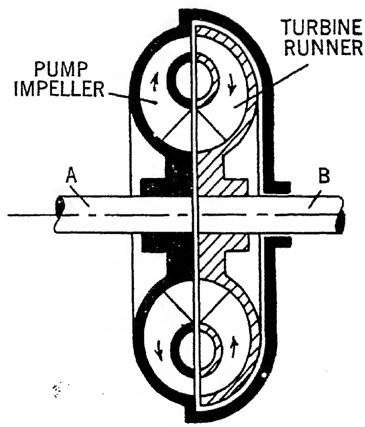


Fig. 215. Fluid coupling.

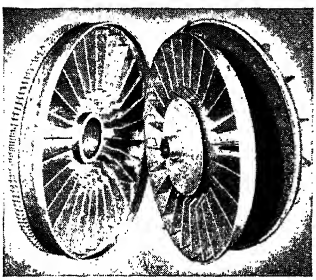
Two rotating shafts could be coupled by an arrangement consisting of a separate pump and a separate turbine joined by intermediate piping. The fluid coupling, however, provides a considerable saving in weight and space because of its concentric arrangement of impeller and runner. Further, the fluid coupling eliminates the friction loss in the intermediate piping.

Since there are no torque reacting elements in the fluid coupling besides the impeller and runner, under steady operating conditions the output torque (shaft B) always equals the input torque (shaft A)—hence the term *fluid coupling*. The speed of shaft A always exceeds that of shaft B. At the beginning of operation, shaft A rotates while shaft B does not; the so-called *slip* is 100 per cent. At rated speeds the slip may be reduced to from 1 to 4 per cent. At normal speeds and loads, the efficiency of a fluid coupling is high, and may be 96 to 99 per cent.

The acceleration of the output shaft is smooth under all conditions of operation. The load may be stalled completely without stalling the driver.

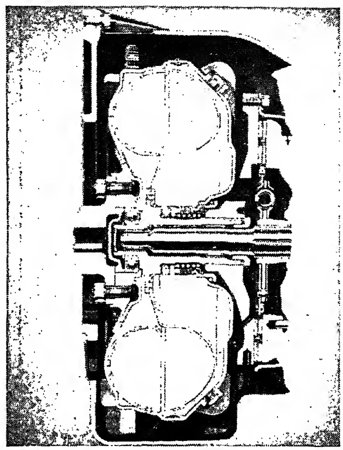
Torsional vibrations or shocks on either shaft of the fluid coupling are damped out by the fluid. In some applications it is very important to eliminate the transmission of such vibrations or shocks. One of the first commercial uses of fluid couplings was in the drives from high-speed Diesel engines to propeller shafts on marine vessels. With no fluid coupling there was considerable wear on the reduction gears (between engines and propeller shaft) because of the engine-shaft torsional vibrations. The added expense of the fluid coupling was offset by the protection to the gears.

Fluid couplings varying in size from 1 to 36,000 horsepower have been built and successfully operated. Fluid couplings are employed in the



Courtesy of the Chrysler Corporation

FIG. 216. Chrysler automotive fluid coupling.



automotive, railroad, marine, and oil-equipment fields, and in countless industrial applications with both electric-motor and internal-combustion engine drives. The fluid coupling does not replace the transmission in the usual automotive application. The extra cost of the coupling, plus the power loss through it, are offset by such factors as reduction in wear on the parts behind the coupling, reduction of vibrations, and improved operation of the vehicle. Figure 216 shows some details of the Chrysler automotive fluid coupling.

157. Fluid torque converters

It is to be emphasized that a fluid torque converter is different from a fluid coupling. As indicated in Fig. 217, the fluid (usually oil) discharged from the pump flows through the turbine runner and a series

of fixed guide vanes. The stationary vanes change the direction of the fluid, thereby making possible a torque and speed transformation. Since the stationary vanes take some reaction (carrying it to the foundation), the turbine (or secondary) torque does not equal the pump (or primary) torque—hence the name *torque converter.* Different arrangements are possible, with several stages of turbine runners and stationary vanes.

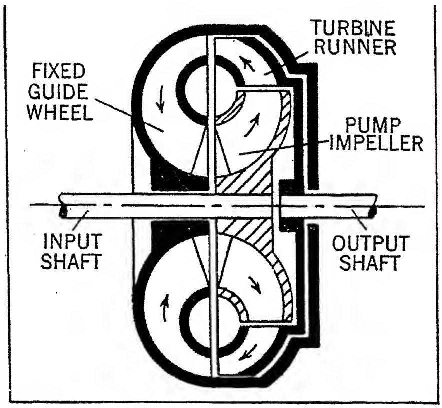


FIG. 217. Torque converter.

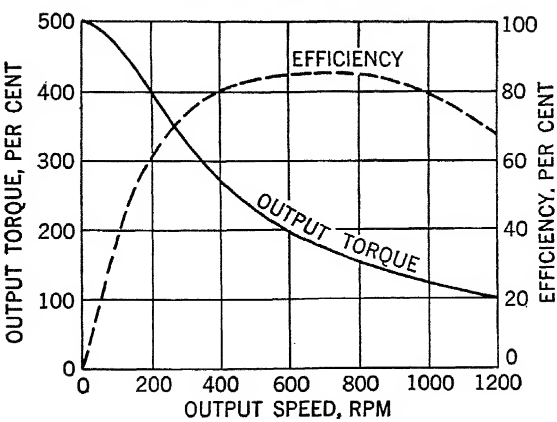


FIG. 218. Characteristics of a torque converter.

The converter provides for smooth starting of the load, and absorbs torsional vibrations and shocks.

In some converters the starting output torque is about five times the input torque. In general, the efficiency of a torque converter is high at low speeds. Converters have been built with peak efficiencies around

85 to 87 per cent. Some typical characteristics of a torque converter are illustrated in Fig. 218. The decreasing torque-speed characteristic is an advantage for the starting and acceleration of heavy loads. If the output shaft of the converter is stalled during operation, owing to extreme loads, the driver will not stall, but will continue running. In some units the speed of the driver is practically constant regardless of the speed of the driven unit.

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PROBLEMS

167. In an industrial process, water is available at 500 pounds per square inch. It is proposed to use this water to operate a turbine at 1750 revolutions per minute against a back pressure of 30 pounds per square inch. What type of turbine would you suggest if it is estimated that a power output of 250 horsepower could be obtained?

168. By what combination of g and specific weight w could the specific speed of a turbine be multiplied in order to give a dimensionless ratio?

169. A supply of water is available at a head of 48 feet. It is proposed to build a turbine to operate at 75 revolutions per minute to develop 42,000 horsepower. What type of turbine would you suggest?

170. A venturi meter installed in a pipe line of 12-inch diameter has a throat diameter of 6 inches. The pressure at inlet is 20 pounds per square inch gage. At what velocity in the main line will cavitation begin if the fluid is water at 104° Fahrenheit?

171. The relative velocity of the water at a certain point on the blade of a marine propeller is always 3.6 times the velocity of the boat which it propels. If this point is 10 feet below the water surface, what will be the velocity of the boat when cavitation begins at the propeller, if the water temperature is 68° Fahrenheit?

CHAPTER 18

Fluid Power and Control Systems

A wide variety of appliances employ a fluid, like oil or air, for power transmission or control. A fluid system has particular features, such as flexibility and capacity to cushion forces, which are desirable or essential in meeting certain service requirements. It is a relatively easy matter to connect one mechanical element with another by fluid conduits. Many functions can be performed by a simple manipulation of valves or pumps. Numerous complicated systems have been devised; many of these are essentially combinations of simple circuits. A study of these simple or basic systems is helpful in analyzing and judging the more complicated circuits.

158. Fluid press

The fluid press, as illustrated in Fig. 219, is probably the simplest type of fluid system. Let F_1 be the force on piston 1, A_1 the area of piston 1, F_2 the force on piston 2, and A_2 the area of piston 2. If friction in the piston guides is neglected, then the pressure p in the fluid is

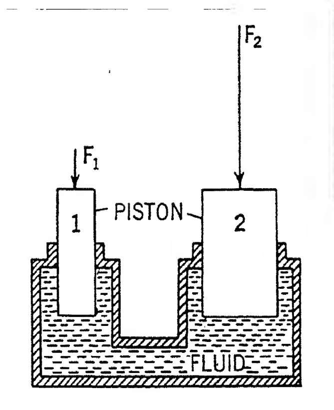


FIG. 219. Fluid press.

$$p = \frac{F_1}{A_1} = \frac{F_2}{A_2}.$$

Since $F_2 = (A_2/A_1)F_1$, a small force on piston 1 can balance a large force at piston 2 if the ratio A_2/A_1 is large. In principle the fluid press is analogous to the mechanical lever.

Figure 220 illustrates an application. The pump develops a pressure to move the actuating piston. The actuating piston in turn moves the wing flap. Cylinder A is commonly called an *actuating cylinder*. The cylinder in Fig. 220 is single acting, that is, the fluid acts only on one side of the piston. The fluid acts on both sides of the piston in a double-acting unit. A spring on the left side of the actuating piston could be included to return the piston to the right. Fluid from the pump could

be used for returning the piston by adding a return line; a *selector* valve of some sort would then be necessary in order to direct the fluid to the side of the piston desired.

A hand pump is suitable in some applications; the jack or hoist on the familiar barber's chair is one illustration. In other applications an engine-driven or motor-driven pump is necessary. Hand pumps are

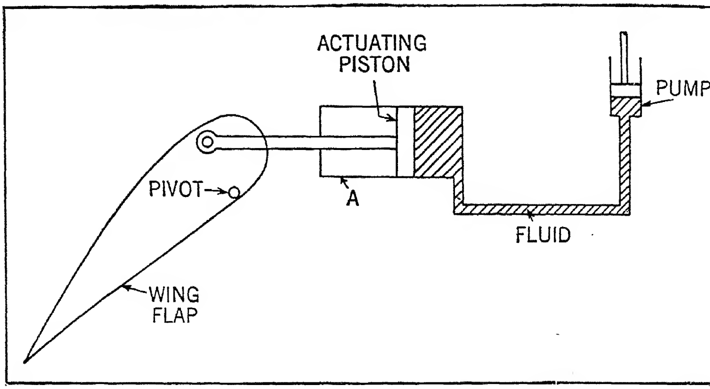


FIG. 220. Simple fluid system.

incorporated in aircraft systems, for example, for emergencies in which the engine-driven pump fails.

159. Simple circuits

There are essentially three main parts in the usual simple fluid system: (1) a fluid pump; (2) a cylinder having an actuating piston driven by the fluid; and (3) piping and valve equipment to control the flow of the fluid. The actuating piston can be given any movement required for power or control. Straight-line reciprocating motion is commonly required. Rotary motion of the driven unit can be accomplished with various forms of fluid motors.

The variable-delivery pump A in Fig. 221 forces some liquid (like oil) to move the actuating piston B. The piston B may be connected to some element where a force is required, as in a press, hoist, jack, or feeding table on a machine tool. The usual pump is of the positive-displacement type; variable delivery can be obtained by using a reciprocating pump with adjustable stroke or adjustable speed. Figure 221 shows the pump pushing the piston outward; a valve would be necessary to interchange the discharge and return pipes for the in-stroke.

For a certain pump speed there will be a certain volumetric rate of

discharge of the liquid practically independent of the load. This rate of pump discharge determines the speed of piston B. The rate at which the piston moves will never exceed the rate corresponding to the pump discharge. The pressure developed by the pump, on the other hand,

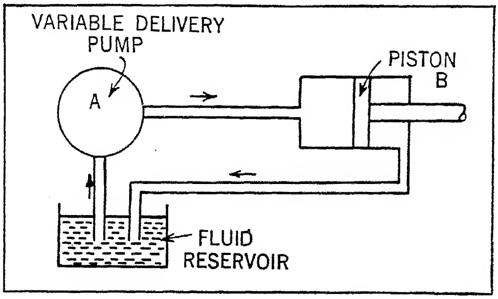


FIG. 221. Simple closed circuit.

depends upon the load or the resistance to be overcome. For example, the load in a grinding machine may be low, whereas the resistance in a boring mill may be high. The speed of piston B can be changed by varying the pump discharge. The type of fluid transmission shown in Fig. 221 gives a sensitive and easy control of the driven unit.

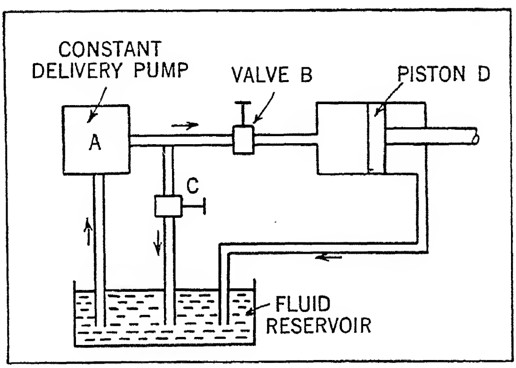


FIG. 222. Fluid system with a by-pass.

Figure 222 shows a system with a constant-delivery pump; the pump may be an inexpensive gear or vane pump. The speed of the actuating piston D is adjusted by valve B, which causes some or much of the liquid from the pump to by-pass through valve C (which may be a relief valve).

160. Some combination circuits

If the speed control required on several actuating pistons is proportional and simultaneous, then the pistons can be arranged in a series

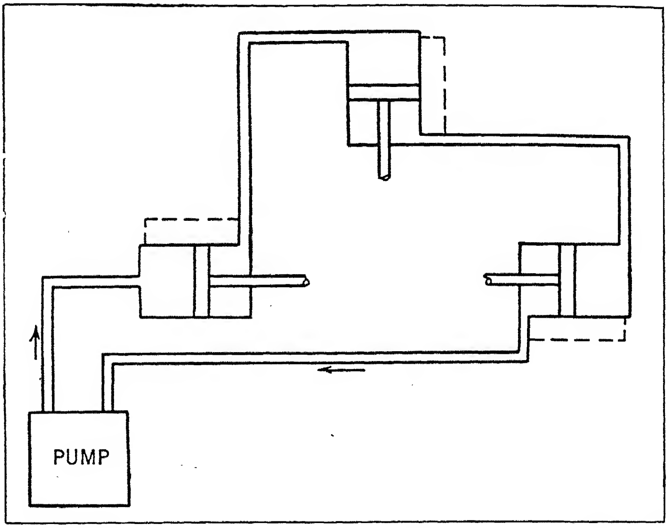


FIG. 223. Circuit with three cylinders in series.

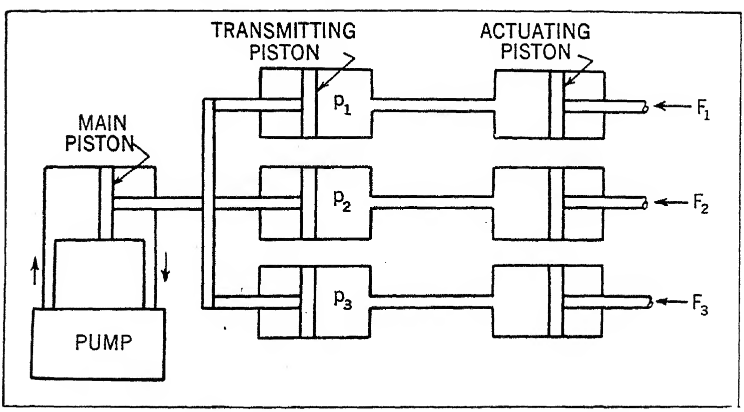


FIG. 224. Multiple-piston system.

closed circuit with a single pump, as shown in Fig. 223. The liquid supplied to the head end of each cylinder after the first comes from the rod end of the preceding cylinder. If leakage is neglected, the volu-

metric displacement of each cylinder equals that of the positive-displacement pump. The cylinder diameters can be arranged to meet the speed requirements of the actuating pistons. If one piston A is to move twice as fast as another piston B, cylinder A must have one-half the volume of B. The dotted lines in Fig. 223 indicate pipe lines with relief valves for coordinating the operation of the separate pistons with each other. This series system divides the total pump working pressure into as many parts as there are cylinders.

The main piston in Fig. 224 is driven by one positive-displacement pump. The main piston rod is fastened to a single crosshead which drives three separate transmitting pistons. Each transmitting piston, in turn, acts as a pump in forcing fluid to drive another piston, an actuating piston. The strokes of all the transmitting pistons are the same. The diameters of the transmitting and actuating cylinders can be arranged to give any desired stroke to each actuating piston. The speed of each actuating piston varies inversely as its piston area. The pressures p_1, p_2, and p_3 each vary directly with the resistances F_1, F_2, and F_3 respectively, and inversely as the piston areas.

161. Accumulator and intensifier

Some systems include an *accumulator*, a device for storing energy. An air accumulator is simply a closed vessel partly filled with air, and

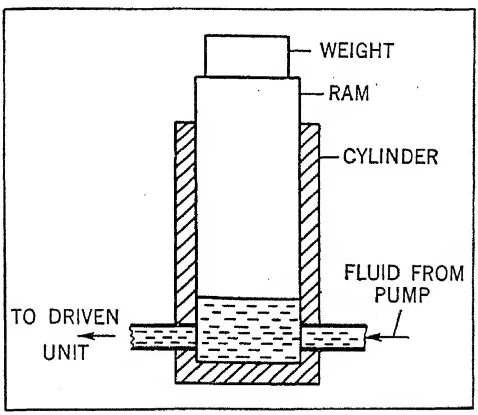


Fig. 225. Liquid accumulator.

into which the pump, which is supplying fluid power to a unit, delivers fluid while the unit is not working at full capacity. Such an air chamber or cushion is commonly found on liquid reciprocating pumps.

Figure 225 illustrates a liquid accumulator. Liquid is admitted to the cylinder; the weight (or spring) is so adjusted that the ram rises when

the fluid pressure reaches a certain amount. The prime object of the accumulator is to permit fluid machines (such as presses and lifts) which are supplied with power from a pump to work for a short time at a greater rate than the pump can supply energy. For a directly connected pump

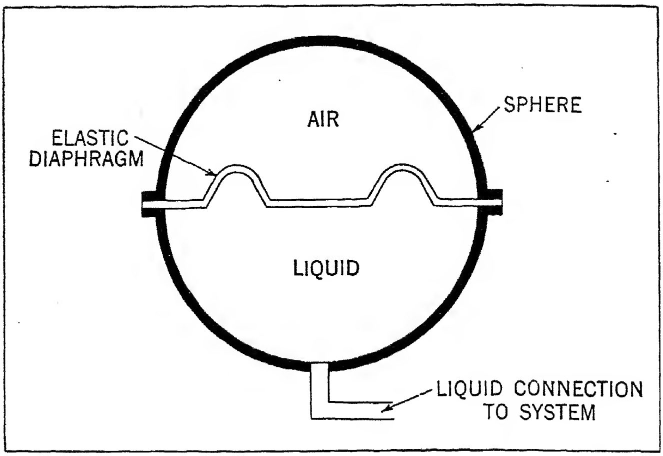


FIG. 226. Spherical accumulator with elastic diaphragm.

and unit, the rate at which the pump can supply energy must equal the rate at which the unit is to work (together with the rate at which friction energy is lost); the pump must be of such a capacity as to supply energy at the greatest rate required by the driven unit. With an accumulator, however, the pump can be kept working steadily during the time when

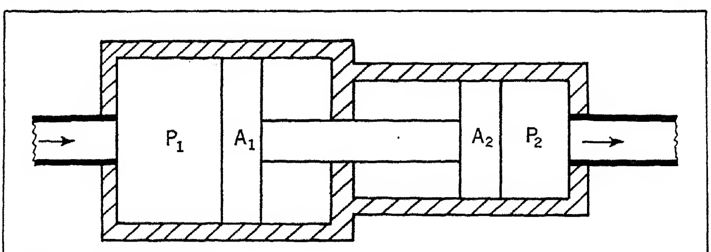
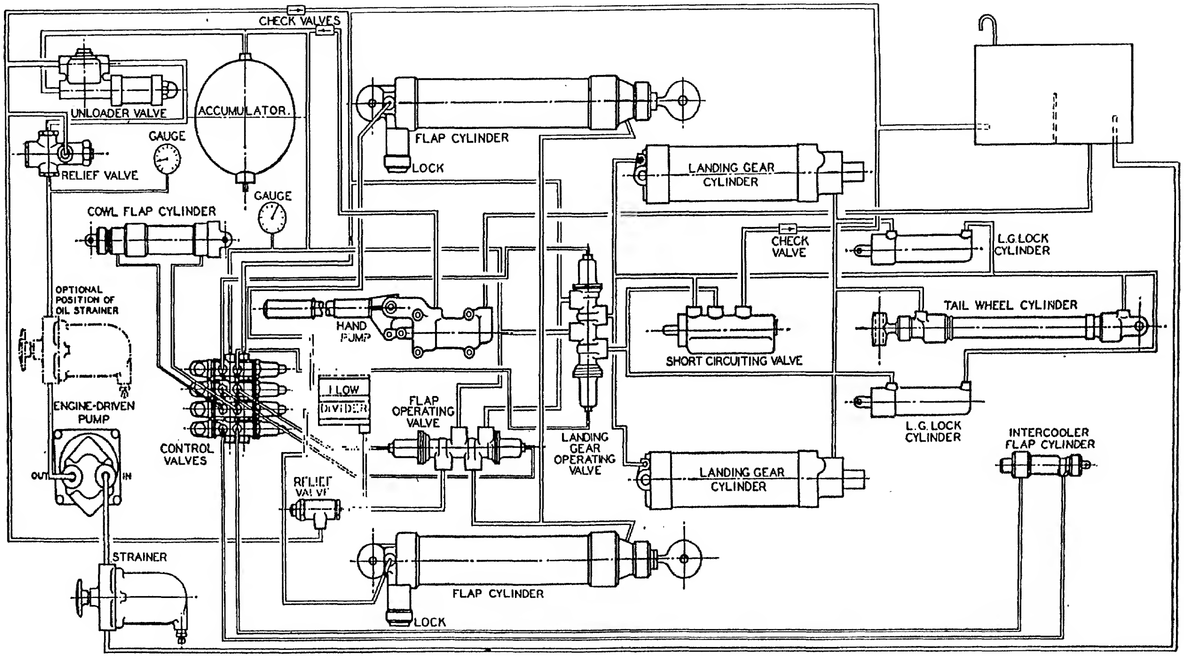


FIG. 227. Intensifier.

the demand is less than the pump supply, and the excess energy stored in the accumulator. The accumulator also works as a pressure regulator; it serves to damp out pressure surges and shocks in the system.

Figure 226 shows the essential features of one type of accumulator. A metal sphere is divided into two hemispheres by an elastic diaphragm.



Courtesy of Air Associates, Inc.

FIG. 228. Aircraft fluid system.

When energy from the accumulator is required, the air pushes the diaphragm, and the liquid moves into the system.

Frequently an *intensifier*, as indicated in Fig. 227, is included in a system, to increase the fluid pressure above that available from a pump. The inlet pressure p_1 acting over the area A_1 develops a force which is transmitted to the second piston. If the area A_2 is less than A_1, the pressure p_2 is greater than p_1.

162. Characteristics of fluid systems

The simple devices discussed in the foregoing articles can be combined in numerous ways to meet different service requirements. A wide variety of combinations is found in practice. Oil, air, water, and mercury are some of the fluids employed. Oil is commonly used; for one thing, it has lubricating properties. Various mineral oils have proved satisfactory in some systems. A mixture of alcohol and castor oil is used in certain applications. Sometimes it is undesirable to use air because of its compressibility. The lines for conveying the fluid may be copper tubing, aluminum alloy tubing, steel tubing, flexible metal hose, or reinforced rubber hose.

Various advantages of fluid power transmission and control systems could be cited. The use of a fluid results in cushioned application of forces. The motion of an actuating piston can be reversed quickly, with practically no shock. Straight-line or rotary motion frequently can be conveniently accomplished at any desired point of application. A fluid system can provide great flexibility in speed control, and can give speed changes in very small steps. Pressure-relief valves and other similar devices can be arranged to provide capacity to stall against overloads without damage to mechanical elements. Control, both hand and automatic, of all movements can be simple and efficient, and can be centralized. In general, fluid systems have relatively few moving parts. Reliability and low upkeep cost result if the wearing parts are relatively few.

Fluid systems are found on milling machines, lathes, drill presses, forging presses, broaches, planers, shapers, grinders, and other machine tools. Some feel that the general trend will follow the use of fluid units for practically all metal-removing machines. In one interesting test a fluid operated shaper showed practically twice the number of strokes per unit time as a mechanical shaper for the same cutting speed. Doubling the productive capacity by using a fluid drive is an attractive feature.

Fluid units are also found on military and naval equipment, road-making machinery, snowplows, and paper mills. Fluid systems are used on aircraft, to operate wing flaps, retractable landing gear, gun turrets, and bomb-bay doors. Figure 228 illustrates an aircraft fluid system.

CHAPTER 19

Mathematical Study of Fluid Motion

Some powerful methods of attack have been developed by many investigators working in the field of theoretical hydrodynamics, or the mathematical theory of fluid motion. In many of the treatments certain assumptions are first made, and then the analysis is carried on with formality, rigor, and elegance. It is to be kept in mind that mathematics is simply a language. A rigorous analysis is only as accurate and reliable as the original assumptions.

At first thought it might appear that some of the theoretical analyses are primarily exercises in mathematical discipline. As such they can be of value to the student. Recent developments, however, have shown fair agreement between purely mathematical results and experimental data for a number of different flow phenomena. The laminar flow in circular pipes and the laminar flow in thick-film lubricated bearings might be cited as examples in which theoretical predictions agree fairly well with physical measurements. The flows around lifting-vane sections of modern design and around streamlined bodies are other examples. A mathematical analysis may be an economical and valuable method for discovering and understanding certain trends which are not apparent from a purely physical study.

One of the first questions that arises in making an analysis is this: What fundamental equations are available for use? Some of these fundamental relations which have been presented in the foregoing chapters of this book are: the equation of state for certain gases; the equation of continuity; the general energy equation for steady flow; the momentum relation; and the relation between shear stress, viscosity, and velocity gradient. Some of these relations have been presented in a simplified form, primarily to answer practical problems, and to bring out physical aspects with a minimum of complexity.

The purpose of the present chapter is to discuss very briefly some of the fundamental relations useful in making a mathematical analysis of fluid motion. The interested student can find abundant material in the reference literature, particularly in the works of Lamb, Prandtl and Tietjens, Durand, Dryden, Murnaghan, and Bateman (see references at the end of this chapter). Some problems can be solved by examining only the geometry of motion without inquiring about the forces causing the motion (kinematics), whereas other problems require an investigation

of the forces acting (dynamics). This chapter will be restricted to the flow of an incompressible fluid, in other words, a fluid whose density ρ is constant.

KINEMATICS

163. Methods of describing motion

Two methods of representing fluid motion have been devised. The so-called Lagrangian method (after Lagrange) refers to a description of the behavior of individual fluid particles. A description of the paths of individual fluid particles would be an example of the Lagrangian method. Similar descriptions could be given for velocities, accelerations, and other characteristics. The so-called Eulerian method (after Euler) refers to a description of the velocity, pressure, and other characteristics at certain points or a section in the fluid. The Eulerian method is the

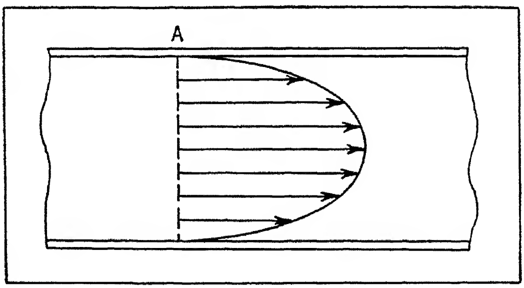


FIG. 229. Example of Eulerian method of description.

more common and familiar. For example, consider the flow in a pipe, as shown in Fig. 229; the velocity distribution at a certain section A is given. Different fluid particles pass this section. It is not known just *which particles* pass this section, but the description gives the velocity at various points at the particular station or section.

Let x, y, and z represent the space coordinates, and t time. Let V represent the resultant velocity at any point in a body of fluid, u the x-component of this resultant velocity, v the y-component, and w the z-component. This notation is illustrated in Fig. 230. As an example of the Eulerian method, the velocity can be expressed symbolically as

$$\begin{aligned} u &= f_1(x, y, z, t), \\ v &= f_2(x, y, z, t), \\ w &= f_3(x, y, z, t). \end{aligned} \tag{251}$$

The first relation in Equation (251) can be stated in words as: the velocity component u is some function of the space coordinates x, y, z, and the

time t. Equation (251) is a general statement of the velocity relation. If the flow is steady, then the velocity at a point is not a function of time, and the foregoing relations become

$$\begin{aligned} u &= f_1(x, y, z), \\ v &= f_2(x, y, z), \\ w &= f_3(x, y, z). \end{aligned} \tag{252}$$

Similar expressions could be written for acceleration, pressure, and other flow features.

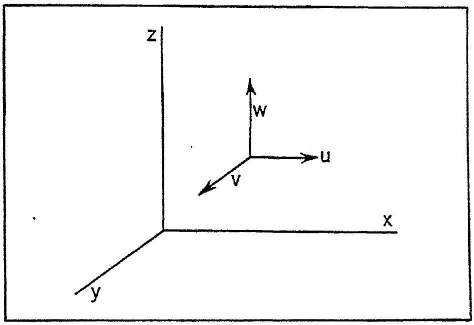


FIG. 230. Coordinate system.

164. Velocity and acceleration

In the most general case of fluid movement the resultant velocity V is a function of both the distance s along a streamline and the time t, that is,

$$V = f(s, t). \tag{253}$$

The velocity changes from point to point in space in one instant of time, and the velocity also changes from moment to moment of time at any one point in space. Thus

$$dV = \frac{\partial V}{\partial s} ds + \frac{\partial V}{\partial t} dt. \tag{254}$$

It is to be recalled that the partial differential quotient is taken with respect to one variable only, keeping all other variables constant. For example, $\partial V/\partial s$ is the derivative of V with respect to s only, with time constant.

Acceleration is the time rate of change of velocity. Thus the total acceleration dV/dt is

$$\frac{dV}{dt} = \frac{\partial V}{\partial s}\frac{ds}{dt} + \frac{\partial V}{\partial t} = V\frac{\partial V}{\partial s} + \frac{\partial V}{\partial t}. \tag{255}$$

The term $\partial V/\partial t$ is frequently called the *local* acceleration, and the term $V(\partial V/\partial s)$ the *convective* acceleration. The local acceleration represents the change in the local velocity with time at a fixed point. If the flow is steady, then $\partial V/\partial t$ equals zero.

Velocity and acceleration are vector quantities; each has both a magnitude and a direction. In some problems it is convenient to deal with components of the resultant vector. Differentiation of Equation (251) gives the corresponding components of the resultant or total acceleration:

$$\frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t},$$
$$\frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t},$$
$$\frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}. \tag{256}$$

For steady motion,

$$\frac{\partial u}{\partial t} = 0, \qquad \frac{\partial v}{\partial t} = 0, \qquad \frac{\partial w}{\partial t} = 0.$$

165. Continuity equation in three dimensions

In Chapter 3 the equation of continuity was given for steady flow in one dimension. The equation of continuity for the steady flow of an

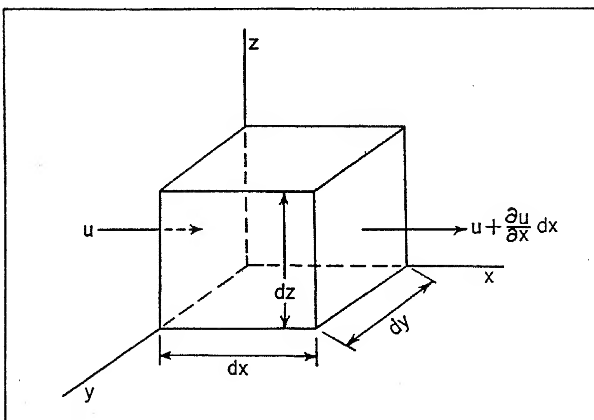


FIG. 231. Infinitesimal parallelepiped in a body of fluid.

incompressible fluid in three dimensions will now be derived. Figure 231 shows an infinitesimal parallelepiped in a body of fluid; the sides have the lengths *dx*, *dy*, and *dz*. Consider first the volume of flow per

unit time in the x-direction. The volume of fluid entering the left per unit time is $u\,dy\,dz$. The velocity at the right face is $u + (\partial u/\partial x)\,dx$. Thus the volume of fluid leaving the right face per unit time is $[u + (\partial u/\partial x)\,dx]\,dy\,dz$. Then the excess volume of fluid flowing out per unit time is

$$\frac{\partial u}{\partial x}\,dx\,dy\,dz.$$

Similarly, expressions for the excess volume per unit time can be written for the y- and z-directions, to give

$$\frac{\partial v}{\partial y}\,dy\,dz\,dx, \qquad \frac{\partial w}{\partial z}\,dz\,dx\,dy.$$

The flow of fluid across the closed boundary surface from without inward must equal that from within outward. If inflow is taken as positive and outflow as negative, then the total volume rate of flow across the closed boundary must be zero. Thus the equation of continuity in three dimensions becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{257}$$

DYNAMICS

166. Relation between force and acceleration

The foregoing relations are kinematic; that is, they refer to the geometry of motion without regard to the forces causing that motion. The next relation to be developed is simply a statement in mathematical form that the resultant force acting on a small element of fluid equals the product of the mass of the element and the acceleration produced. Note that both force and acceleration are vector quantities. The following treatment is limited to the flow of an incompressible fluid with no friction or viscous forces acting (the dynamic viscosity being equal to zero).

Figure 232 shows an infinitesimal parallelepiped in a body of fluid; the sides have the lengths dx, dy, and dz. Consider first the forces acting on this element in the x-direction. The pressure on the left face is p; the pressure force on the left face is $p\,dy\,dz$. The pressure on the right face is $p + \frac{\partial p}{\partial x}\,dx$; the pressure force on the right face is

$$\left(p + \frac{\partial p}{\partial x}\,dx\right)dy\,dz.$$

The net pressure force acting on the element in the positive direction of x is

$$-\frac{\partial p}{\partial x}\,dx\,dy\,dz.$$

Assume that some force other than pressure is acting on the fluid; such a force might be a gravity force. The attraction of the earth is an example of a so-called *body* force. A body force is proportional to the volume or mass, whereas a *surface* force, such as pressure, is proportional to the area. Let P, Q, and R represent the x-, y-, and z-com-

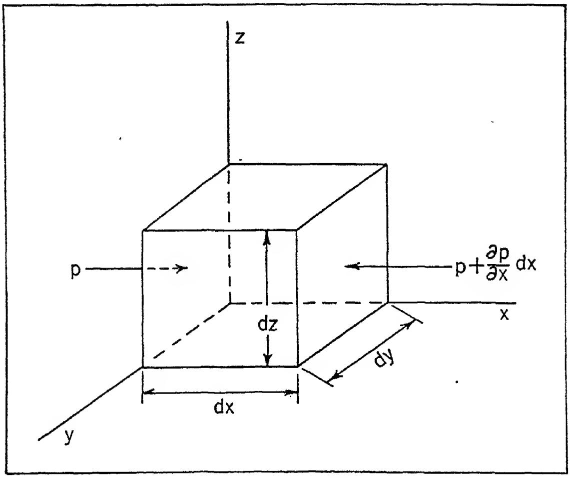


FIG. 232. Infinitesimal parallelepiped of fluid.

ponents, respectively, of the body force per unit mass. Then the total force acting on the element in the positive x-direction is

$$\left(P\rho - \frac{\partial p}{\partial x}\right) dx\,dy\,dz.$$

Since force equals mass times acceleration, the dynamical relation for the x-direction becomes

$$\left(P\rho - \frac{\partial p}{\partial x}\right) dx\,dy\,dz = \rho\,dx\,dy\,dz\,\frac{du}{dt},$$

$$P\rho - \frac{\partial p}{\partial x} = \rho\,\frac{du}{dt}, \tag{258}$$

where du/dt is the x-component of the total acceleration. Equations for the y- and z-directions can be obtained in a similar fashion. The final set of dynamical equations for the three directions is thus

$$P\rho - \frac{\partial p}{\partial x} = \rho \frac{du}{dt},$$
$$Q\rho - \frac{\partial p}{\partial y} = \rho \frac{dv}{dt},$$
$$R\rho - \frac{\partial p}{\partial z} = \rho \frac{dw}{dt}. \tag{259}$$

Equations (259) are sometimes called Euler's hydrodynamical equations. The total acceleration in each relation in Equation (259) can be expressed in the form given by Equation (256).

TWO-DIMENSIONAL MOTION

167. General remarks

The remainder of this chapter will be devoted to the two-dimensional steady flow of a frictionless, incompressible fluid. In two-dimensional motion the flow is identical in parallel planes. The xy-plane will be taken as the plane of flow.

It was pointed out in Chapter 3 that a general problem is to map out the velocity distribution in the field of a flowing fluid. If the velocity at each point in the fluid is known, an application of the energy equation yields the pressure at each point. The total force acting on a body in a fluid can be determined from the pressure distribution around the body. Chapter 3 outlined briefly how the velocity distribution or streamline pattern can be obtained graphically by combining or superimposing simple types of flow. The following articles will discuss a method for determining the velocity field analytically. The general method of approach employed in this fluid flow problem is also used in other branches of applied physics, as in rigid-body mechanics, elasticity, electricity, and magnetism.

So-called *potential functions* have been devised for expediting the analytical investigation of velocity fields. One type of function is called a *velocity potential.* Let ϕ (Greek letter phi) represent the velocity potential; ϕ is some function of x and y, and is defined such that

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}. \tag{260}$$

Differentiation with respect to two variables is independent of the order of differentiation, that is,

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}.$$

Thus Equation (261) imposes a restriction upon the type of flow which can be described by the potential function ϕ. This restriction is

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}. \tag{262}$$

The use of the potential function ϕ is limited to irrotational motion; this type of motion will be discussed further in the next article.

168. Rotational and irrotational motion

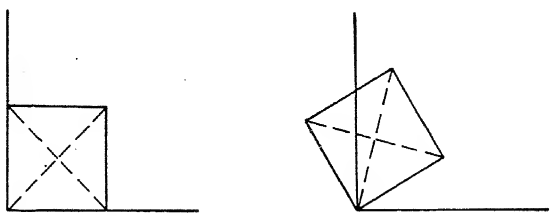
It was pointed out in Article 83 that in so-called rotational motion each infinitesimal particle in the field of flow rotates about its own axis.

(a) (b)

Fig. 233. Translation and linear deformation of a fluid element.

It was also pointed out in Article 83 that in irrotational motion an infinitesimal fluid particle does not rotate about its own axis.

A fluid element may undergo four types of movement: (1) a pure translation; (2) a linear deformation; (3) an angular or shearing deformation; and (4) a rotation. The solid lines in Fig. 233a represent an infinitesimal fluid element at a certain instant. The dotted lines represent the element for a pure translation; there is no change in the lengths



(a) (b)

Fig. 234. Rotation of a fluid element.

of the parallel sides. The dotted lines in Fig. 233b represent a condition after linear deformation has taken place. Figure 234b indicates a rotation of the element from the position initially shown in Fig. 234a. The dotted diagonal lines in Fig. 234 have turned through an angle.

Fig. 235b indicates the position of an element after an angular or shearing deformation has taken place; the element was initially in the position shown in Fig. 235a. During a time interval dt the infinitesimal element line initially along the x-axis has experienced an angular change equal to $\frac{\partial v}{\partial x}\,dt$, whereas the element line initially along the y-axis has

experienced an angular change equal to $\frac{\partial u}{\partial y}\,dt$. The mean *rotation* of an element is equal to the average, or

$$\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

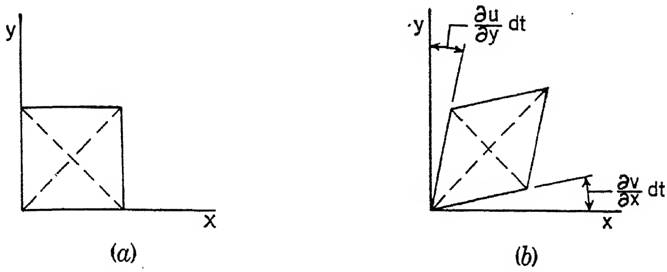


FIG. 235. Angular or shearing deformation of a fluid element.

If $\partial v/\partial x = \partial u/\partial y$, then the rotation is zero. The sides of the element may undergo an angular deformation, but the diagonals of the infinitesimal element have not rotated, and the motion is classed as irrotational.

169. Velocity potential

The remainder of this chapter will be devoted to irrotational motion. Some examples and preliminary features will be given first, before investigating the technically important problem of flow around a cylinder.

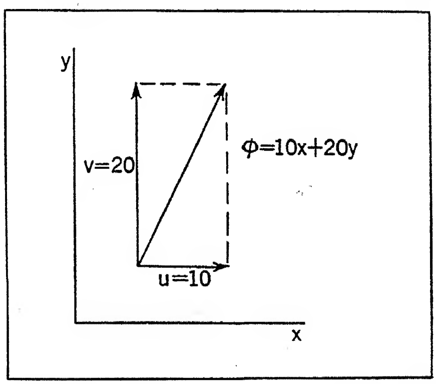


FIG. 236. Vector addition of velocities.

It was pointed out that the velocity potential ϕ is a function of x and y. The feet and second units will be employed in presenting a few examples. If the velocity potential for a certain flow is $\phi_1 = 10x$, then

$u = 10$ feet per second ($u = \partial\phi/\partial x$). The velocity is 10 feet per second in the x-direction for all points in the field of flow. If the velocity potential for another flow is $\phi_2 = 20y$, then $v = 20$ feet per second. The velocity at all points is 20 feet per second in the y-direction. These two flows can be superimposed or combined mathematically by simply adding the two potential functions. The *algebraic* addition of two potential functions is equivalent to a vector addition of the velocities. The potential function $\phi = \phi_1 + \phi_2 = 10x + 20y$ represents the combined flow pattern, and is illustrated in Fig. 236 for one point. This simple example illustrates the convenience of potential functions.

170. Velocity potential for source, sink, and doublet

Certain simple types of flow can be combined to give various resultant streamline patterns. The streamlines can be regarded as solid boundaries, since no fluid crosses a streamline. A closed streamline, then, can be replaced by a solid body without affecting the flow. It will be necessary to derive the velocity potentials for a few simple flows before investigating combined flows.

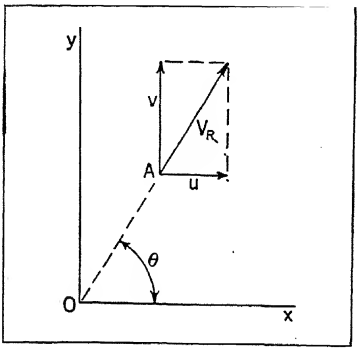


FIG. 237. Notation for a source flow.

Figure 237 shows the notation for the flow due to a source placed at the origin 0 of the coordinate axes. The coordinates of point A are x and y. The streamlines in a source flow are radial lines. The resultant velocity at point A is V_R, in a radial direction outward. The x- and y-components of the resultant velocity are

$$u = V_R \cos\theta, \qquad v = V_R \sin\theta.$$

Let the radial distance from the origin O to point A be represented by R. Assume a distance equal to unity perpendicular to the plane of flow, and let Q represent the volume of flow per unit time or the *flux* from the source. Then $Q = 2\pi R V_R$. Let ϕ_s represent the velocity potential for the source. Then

$$u = \frac{Q}{2\pi R}\cos\theta = \frac{Q}{2\pi}\left(\frac{x}{x^2 + y^2}\right) = \frac{\partial\phi_s}{\partial x}, \tag{263}$$

$$v = \frac{Q}{2\pi R}\sin\theta = \frac{Q}{2\pi}\left(\frac{y}{x^2 + y^2}\right) = \frac{\partial\phi_s}{\partial y}. \tag{264}$$

Integration of Equations (263) and (264) shows that

$$\phi_s = \frac{Q}{4\pi} \log (x^2 + y^2), \tag{265}$$

which can be checked by differentiation. Since $R^2 = x^2 + y^2$,

$$\phi_s = \frac{Q}{2\pi} \log R. \tag{266}$$

The potential function for a sink can be derived in a similar fashion. The velocity potential for a sink is simply the negative of that for a source.

Figure 238 shows the notation which will be used in deriving the potential function for a so-called *doublet*. A source is placed at A, and

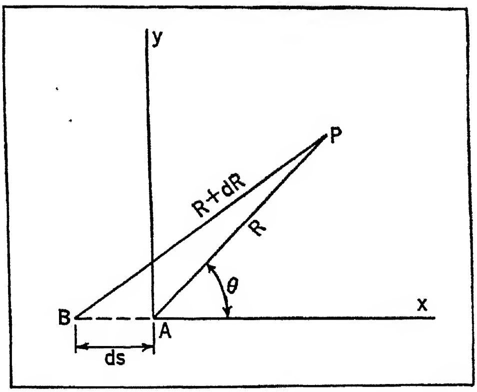


FIG. 238. Notation for a combination of source and sink.

a sink of equal strength (same Q) is placed at B. The distance ds is infinitesimal. The potential function at any point P is the sum of the source and sink potential functions. Thus

$$\phi_D = \frac{Q}{2\pi} \log R - \frac{Q}{2\pi} \log (R + dR),$$

$$\phi_D = -\frac{Q}{2\pi} \log \left[1 + \frac{dR}{R}\right]. \tag{267}$$

Expanding Equation (267) in terms of a power series gives

$$\phi_D = -\frac{Q}{2\pi}\left[\frac{dR}{R} - \frac{1}{2}\left(\frac{dR}{R}\right)^2 \cdots\right]. \tag{268}$$

If infinitesimals of order higher than the first are neglected, then

$$\phi_D = -\frac{Q}{2\pi}\left[\frac{dR}{R}\right] = -\frac{Qds\cos\theta}{2\pi R}. \tag{269}$$

Imagine the distance ds to be continuously decreased while the term $Q/2\pi$ increases so that the product $(Q/2\pi)\,ds$ remains constant. Such a combination of flows at the limit is called a *doublet.* Let $Qds/2\pi = F$; the value of F will be determined in the next article. Then the velocity potential function for the doublet is

$$\phi_D = -\frac{F \cos \theta}{R} = -\frac{Fx}{x^2 + y^2}. \tag{270}$$

171. Flow around a circular cylinder

It was pointed out in Article 28, Chapter 3, that the superposition of a source and a uniform rectilinear flow gives a resultant flow which can be regarded as the flow around a half-streamline body.

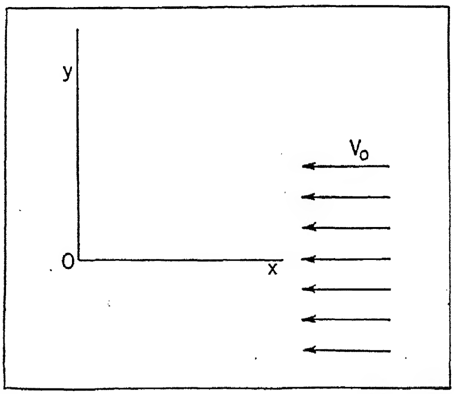


FIG. 239. Doublet at point O with uniform rectilinear flow.

Assume that a doublet of the type discussed in the foregoing article is placed at the origin O of the coordinate axes shown in Fig. 239. A uniform rectilinear flow with a velocity $-V_0$ along the x-axis is superposed on this doublet flow. The resulting streamline pattern is shown in Fig. 240. It is to be noted that one streamline is a circle with its center at the origin of the coordinate axes. A circular cylinder or body can be substituted for this particular closed streamline without influencing the flow; the combination of a doublet flow and a rectilinear flow thus gives the flow around a circular cylinder.

The velocity potential for the rectilinear flow is simply $-V_0x$. The total velocity potential for the combined flow is

$$\phi = -V_0x - \frac{Fx}{x^2 + y^2}. \tag{271}$$

The velocity in the x-direction at any point in the field of flow is

$$u = \frac{\partial \phi}{\partial x} = -V_0 - \left[\frac{(x^2 + y^2)F - xF(2x)}{(x^2 + y^2)^2}\right]. \tag{272}$$

Let a represent the radius of the circular cylinder. The value of F can be determined by using the condition that $u = 0$ at the point $x = a$, $y = 0$. Making this substitution shows that

$$F = a^2V_0.$$

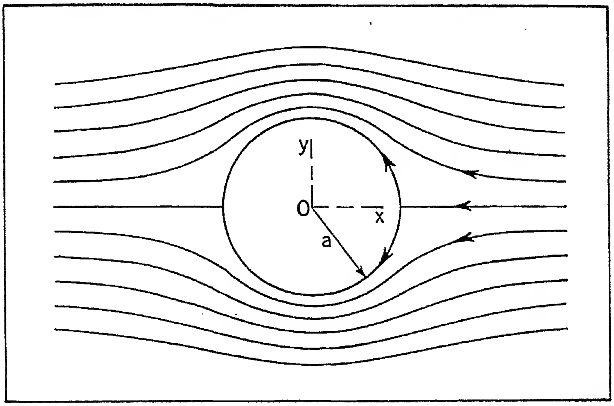


FIG. 240. Resultant flow due to a combination of a doublet and a uniform rectilinear flow.

Substituting the foregoing value of F in Equation (272) gives

$$\phi = -V_0x - \frac{xV_0a^2}{x^2 + y^2}. \tag{273}$$

Since $u = \partial\phi/\partial x$ and $v = \partial\phi/\partial y$, the horizontal and vertical components of the resultant velocity at any point in the field of flow can be found by employing Equation (273). The simple algebraic expression of Equation (273) illustrates the convenience and value of potential functions.

172. Pressure distribution around a cylinder

In order to determine the pressure at points along the surface of the cylinder it is necessary to find the resultant velocity V_s at any point at the surface of the cylinder. Let θ be the angle between the radial line to any point A and the x-axis; the notation is shown in Fig. 241. At the cylinder surface, $x = a \cos \theta$ and $y = a \sin \theta$. The magnitude of the resultant velocity V_s is given by the relation

$$V_s = \sqrt{u_s^2 + v_s^2}, \tag{274}$$

where u_s and v_s are the x- and y-components of the velocity at the cylinder surface; these components can be calculated by means of Equation (273). The final result is simply

$$V_s = 2V_0 \sin \theta. \tag{275}$$

Note that the velocity V_s is a maximum when θ is 90°.

The velocity some distance ahead of the cylinder is V_0. Let the static pressure in the undisturbed stream ahead of the cylinder be represented by p_0. Let p be the static pressure at any point A on the surface

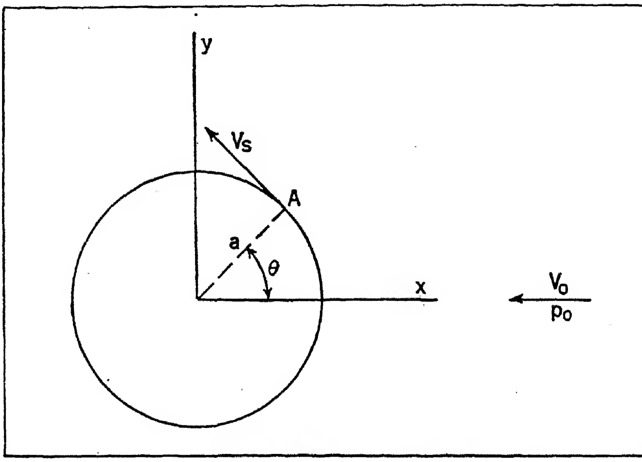


FIG. 241. Notation for cylinder in potential flow.

of the cylinder. Applying the energy equation between a point in the undisturbed stream and the point A gives

$$p + \frac{\rho V_s^2}{2} = p_0 + \frac{\rho V_0^2}{2}. \tag{276}$$

Since $V_s = 2V_0 \sin \theta$, Equation (276) becomes

$$p = p_0 + \frac{\rho V_0^2}{2}[1 - 4 \sin^2 \theta]. \tag{277}$$

Thus the pressure at any point on the surface of the cylinder can be calculated by Equation (277).

173. Flow around a cylinder with a free vortex

The flow around the cylinder discussed in the two foregoing articles is a combination of a doublet and a uniform rectilinear flow. As illus-

trated in Fig. 240, such a flow is symmetrical with respect to the line of undisturbed motion (the x-axis). For one thing, there is no lift on a cylinder in such a flow. A dynamic lift can be developed if the streamline pattern is nonsymmetrical with respect to the line of undisturbed motion. An example of a nonsymmetrical flow is given in Fig. 113. A nonsymmetrical flow can be obtained by adding a free vortex (irrotational motion) to the doublet and rectilinear flow.

The free vortex was discussed in Article 83. Let V_P represent the velocity at the surface of the cylinder due to a counterclockwise free

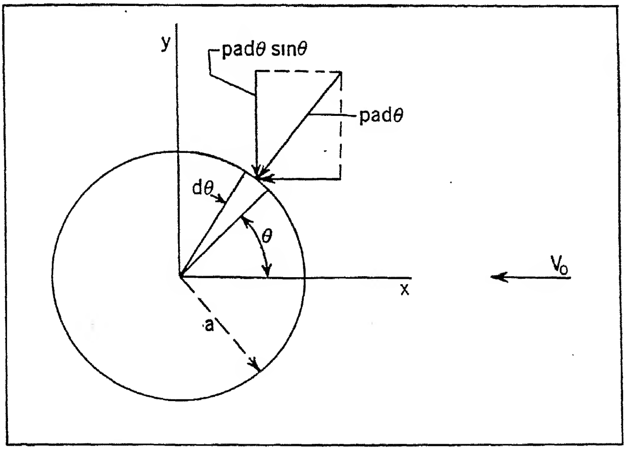


Fig. 242. Notation for determining lift force on a cylinder.

vortex alone. Then the circulation Γ around the cylinder is given by the relation

$$\Gamma = 2\pi a V_P.$$

Thus the magnitude of the resultant or total velocity V_T at the surface of the cylinder is

$$V_T = V_S + V_P = 2V_0 \sin\theta + \frac{\Gamma}{2\pi a}. \tag{278}$$

Applying the energy equation between a point in the undisturbed stream and a point on the cylinder gives

$$p_0 + \tfrac{1}{2}\rho V_0^2 = p + \tfrac{1}{2}\rho V_T^2$$

or

$$p = p_0 + \frac{1}{2}\rho V_0^2 - \frac{\rho}{2}\left[2V_0 \sin\theta + \frac{\Gamma}{2\pi a}\right]^2. \tag{279}$$

The static pressure p at any point on the surface of the cylinder can be calculated by Equation (279).

The next step is to calculate the net lift force at right angles to the undisturbed motion. This lift force can be determined by integrating the pressure forces over the cylinder in the y-direction. The static pressure p acts normal to the surface of the cylinder. Let $d\theta$ be the differential change in the angle θ; the notation is illustrated in Fig. 242. Assume a distance equal to unity perpendicular to the plane of flow. The pressure force on the element of area $a\,d\theta$ is $pa\,d\theta$. The component of this force at right angles to V_0 is $pa\,d\theta \sin\theta$. The lift on the elementary area is $-pa\,d\theta\sin\theta$; the minus sign indicates the direction of the force. The total lift L on the cylinder is

$$L = -\int_0^{2\pi} pa \sin\theta\, d\theta. \tag{280}$$

The relation for p as given by Equation (279) can be substituted in Equation (280), and the resulting expression integrated. The final result is

$$L = \rho\Gamma V_0. \tag{281}$$

The lift force per unit length of cylinder equals the product of density, circulation, and the undisturbed velocity. Equation (281) is the Kutta-Joukowsky relation which was discussed in Article 93.

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PROBLEMS

172. Starting with Equation (274), derive Equation (275).

173. Starting with Equation (280), derive Equation (281).

174. At what points on the surface of a cylinder in potential flow with no circulation is the static pressure p equal to p_0?

175. For the potential flow around a cylinder with no circulation, plot a diagram of the dimensionless ratio $\dfrac{p - p_0}{}$ against θ.

ANSWERS TO PROBLEMS

1. $w = 56.1$ lb. per cu. ft., $\rho = 1.742$ slugs per cu. ft., $w = 845$ lb. per cu. ft., $\rho = 26.25$ slugs per cu. ft.
2. $\rho = 0.7771$ slugs per cu. ft., $w = 24.97$ lb. per cu. ft.
3. $w = 0.825$ lb. per cu. ft., $\rho = 0.0256$ slugs per cu. ft.
4. $w = 0.321$ lb. per cu. ft., $v = 3.12$ cu. ft. per lb., $\rho = 0.00998$ slugs per cu. ft.
5. 96.1 ft. per °C.
6. 14.1 lb. per sq. in. absolute.
7. $v = 184.6$ cu. ft. per lb., $v = 139.5$ cu. ft. per lb.
8. 25.9 tons.
9. 57.7 ft., 51.1 in.
10. 132.3 ft.
11. 5.25 ft. in each bell.
✓ **12.** 16.65 ft., 79 cu. ft.
13. 14.14 in.
14. 15.11 lb. per sq. in., 0.00244 slugs per cu. ft.
15. 8.22 lb. per sq. in.
16. 0.0538 lb. per sq. in. below atmospheric pressure.
17. -0.405 lb. per sq. in.
18. 17.82 lb. per sq. in.
19. 4710 lb., 3930 lb., 2950 lb., 3.33 ft. below free surface.
20. 1635 lb., 8500 lb., 3.0 ft. below free surface.
21. 4915 lb.
22. 40.7 lb.
23. 34,500 lb., 0.346 ft. below center of disk in the plane of the disk.
24. 5820 lb.
25. 716 lb.
26. 0.449 cu. ft., 3.215.
27. 11,700 lb.
28. 3.21 ft., 369 tons.
29. 0.0833.
30. 67.0 tons, 1,850,000 cu. ft.
31. 6.41 tons.
34. 1,027 feet.
35. (a) 8.34 lb. per sq. in., (b) 9.53 lb. per sq. in., and (c) 9.26 lb. per sq. in.
36. Up.
37. 1.882 ft., 4.118 ft., 442 lb., 2117 lb.
38. 27.8 radians per second, 1.64 lb. per sq. in. gage.
40. 12-in. section $V = 3.75$ ft. per sec., 10-in. section $V = 5.4$ ft. per sec.
41. 8-in. section $V = 2.81$ ft. per sec., 14-in. section $V = 1.17$ ft. per sec.
42. 0.636 ft.
43. 0.1136 lb. per sq. in.
44. 47.6 lb. per min.
45. 2670 ft. per sec.
46. 30.4 hp.
47. 50.12 B.t.u. added per lb.

48. 18.6 B.t.u. per lb.
49. 650,000 B.t.u. per hr.
51. Work done on air is 58,000 ft.-lb. per lb. of air.
52. 5.06 lb. per sq. ft.
53. $\mu = 3.86 \times 10^{-5}$ slug per ft.-sec., $\nu = 1.904 \times 10^{-5}$ ft.^{2} per sec., 0.01769 stokes.
54. 0.0337 slug per ft.-sec.
55. 0.0338 lb., 44.8 lb.
56. $c = \text{constant}\ \sqrt{p/\rho}$.
57. $\frac{R}{\rho V^2 l^2} = \phi\left(\frac{\rho V l}{\mu}\right)$.
58. $\frac{R}{\rho V^2 b^2} = \phi\left[\frac{h}{b}, \frac{\mu}{\rho V b}\right]$.
59. $\frac{Q}{D^3 N} = \phi\left[\frac{\rho N D^2}{\mu}, \frac{\mu N D}{S}, \frac{wD}{\mu N}\right.$
60. $\frac{l}{D} = \phi\left(\frac{V_1}{V_2}\right)$.
61. 7.85 ft. per sec.
62. 20 atmospheres.
63. 2.32 revolutions per min.
64. 60 mi. per hr.
65. 218 revolutions per min.
66. 2 mi. per hr.
67. 126 lb. per sq. in.
68. 2.5 lb. per sq. in.
69. 0.956 hp.
70. 1.39 lb. per sq. in.
71. 0.572 lb. per sq. in.
72. 30.4 hp.
73. 7740.
74. 6 in.
75. 41.9 cu. ft. per sec.
76. 28.1 ft. per sec.
78. 1.173 cu. ft. per sec.
79. 0.85 cu. ft. per sec.
80. $C_v = 0.90$, $C_c = 0.654$.
81. 0.0603 cu. ft. per sec.
82. 8.86×10^{-5} slug per ft.-sec.
83. 4.23×10^{-4} slug per ft.-sec.
84. 18.6 lb.
85. 25.2 lb., 327 ft.-lb. per sec.
86. 24.3 lb.
87. 650 lb.
88. 11,250 lb.
89. 75 per cent, 25 per cent.
91. 0.709.
92. 0.015 ft. per sec.
93. 37.6×10^{-4} slug per ft.-sec.
94. Up, up, down.
95. 0.642 lb.

96. 4.81 lb.
97. 4.92 lb., 4010 lb.
98. 13.08 hp., 1.63 hp.
99. 76,300 lb.
100. 89 hp.
101. 145.3 hp.
102. 1150 lb.
103. 1710 lb.
104. 0.842.
105. 3940 lb.
106. $\Gamma = 2\pi r^2\omega$.
108. 202 lb.
109. 0, 33.3 hp.
110. 7.85 ft. per sec., 18°54′.
111. 93.5 per cent, 797 hp.
112. 0.098 hp.
113. 6 hp.
115. 1.12 sec.
116. 17.97 lb. per sq. in. absolute, 17.78 lb. per sq. in. absolute.
117. At A, $N_M = 0.402$; at B, $N_M = 1.07$.
118. 2386 ft. per sec.
119. 1.55 lb. per sec.
125. 340 lb. per sq. in. absolute.
126. 11.6 lb. per sq. in., 12.4 lb. per sq. in.
127. 849 ft.
128. 18.6 lb. per sq. in. absolute, 386 ft.
129. 2335 ft., 12.75 lb. per sec.
130. 33.6 cu. ft. per sec.
131. 0.000084.
132. 980 cu. ft. per sec.
133. $b = 2d$.
134. 3.88 ft.
135. Tranquil.
136. 876 cu. ft. per sec.
140. No.
141. 54,000 ft.-lb. per sec.
142. 21.4 per cent.
143. 2.416×10^{-9}.
145. 54.5.
146. 5.55×10^{-3} slug per ft.-sec., 2.085×10^{-4} slug per ft.-sec.
147. 0.0852 ft.^{2} per sec., 0.000056 ft.^{2} per sec.
148. 28.7.
149. 0.926 hp.
150. 0.00127 in.
151. 1481.
152. Gear oil, S.A.E. 140.
153. 750 revolutions per min.
154. Maximum value of $\dfrac{(p - p_0)}{3\mu \dfrac{V}{h_m}} = \dfrac{l^2/a^2}{4\delta\left(1 - \dfrac{l}{a}\right)}$.
155. 0.77, 3790.
156. 37.1 hp.

157. 1.34 cents.
158. 1218 gal. per min., 87.0 ft.
159. 518 gal. per min., 36.3 ft.
160. 1695.
162. Single-suction centrifugal.
163. Propeller type.
164. 0.42 hp.
165. 81.3 per cent.
166. $\phi\left[\frac{Q}{(gH)^{1/2}D^2}, \frac{ND}{(gH)^{1/2}}, \frac{\mu}{\rho D(gH)^{1/2}}\right] = 0.$
167. Impulse type.
168. $w^{-1/2}g^{-3/4}$.
169. Propeller type.
170. 18.2 ft. per sec.
171. 15.2 ft. per sec.
174. $\theta = 30°, 160°, 210°, 330°$.

Index